# HOMOLOGICAL FEATURES OF RINGS OF CONTINUOUS FUNCTIONS

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1997

*For Laura...a true friend...always...*

## ACKNOWLEDGEMENTS

I wish to thank all of those people who have inspired me to reach beyond my existing abilities and strive for that which I, at one time, could only dream of. The list of names includes many, and you know who you are.

I especially wish to thank Dr. Jorge Martinez. His dedication to teaching his students how to *think*, how to find out *what makes it tick*, has been an unending source of strength and inspiration. I only hope that I might do justice to the skills he passed on to me.

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August 1997

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The following is an investigation into the relationship between the homological properties of the ring of all real valued, continuous functions on a topological space, $C(X)$, and the underlying topological properties of the space $X$. Regarding the weak dimension of $C(X)$, it is shown that, given an $SV$-space, $X$, the inequality $wD(C(X)) \leq rk(X)$ always holds, where $rk(X)$ denotes the rank of the space $X$. With respect to the global dimension of $C(X)$, we prove that, for $X$ a $P$-space, if the tightness of the space $\beta X$ is greater than or equal to $\aleph_n$, then $gD(C(X)) \geq n+1$.

# CHAPTER 1
## Introduction

In the mid-1950's the theory of rings of continuous functions began a rapid development, most notably with the publication of Gillman and Henriksen's *Rings of continuous functions in which every finitely generated ideal is principal.* This paper (which characterized Bézout rings of real valued continuous functions) can be seen as the beginning of a theory of rings of continuous functions, whose aim was to investigate the interaction between topological properties of the space $X$ and algebraic properties of the ring, $C(X)$, of continuous functions from $X$ into $\mathbb{R}$, the ordered field of real numbers.

Following the publication of *Rings of continuous functions in which every finitely generated ideal is principal* the discipline achieved an exceptional degree of clarity and development in the 1960 printing of Gillman and Jerison's *Rings of Continuous Functions.* It was from this point on that the study of rings of continuous functions began in earnest its interaction with disciplines such as commutative algebra, the theory of completely regular spaces, ordered algebraic structures, and even model theory. The mathematics which proceeded from these interactions is rich and varied and includes the following results which we list with appropriate topological and algebraic definitions:

(a) *$X$ is an F-space if and only if $C(X)$ is Bézout.* (1956, [GH2])

We call a space, $X$, an *F-space* if every cozero set of $X$ is $C*$-embedded in $X$. A commutative ring with identity, $A$, is said to be *Bézout* if every finitely

generated ideal of $A$ is principal. We note here that although the result in (a) is actually the definition of an *F-space* given in [GH2], as it is more appropriate for this exposition, we shall use the equivalent definition stated in terms of cozero sets given above.

(b) *$X$ is a P-space if and only if $C(X)$ is von Neumann regular.* (1956, [GH1])

We call a space, $X$, a *P-space* if the topology for the open sets on $X$ is closed under countable intersections. A commutative ring with identity, $A$, is said to be *von Neumann regular* if, for every $a \in A$, there exists an $x \in A$, such that $a^2x = a$.

(c) *$X$ is quasi F if and only if $C(X)$ is Prüfer.* (1992, [MW])

We say a space, $X$, is *quasi-F* if every dense cozero set of $X$ is $C^*$-embedded in $X$. A commutative ring with identity, $A$, is said to be *Prüfer* if every finitely generated ideal which contains a regular element is invertible, relative to the usual Krull product of ideals.

(d) *$X$ is basically disconnected if and only if $C(X)$ is coherent.* (1992, [Ne])

We say a space, $X$, is *basically disconnected* if the closure of every cozero set in $X$ is clopen in $X$. A commutative ring with identity, $A$, is said to be *coherent* if every infinite product of copies of $A$ is flat as an $A$-module.

(e) *$X$ is basically disconnected if and only if every finitely generated ideal of $C(X)$ is projective.* (1978, [Br2]; 1983, [DM])

We remind the reader that an ideal is *projective* if it is a summand of a free $C(X)$-module.

(f) *$X$ is finite if and only if every ideal of $C(X)$ is projective.* (1978, [Br2]; 1983, [DM])

(g) *$X$ is an F-space if and only if $C(X)$ is 1-convex.* (1982, folklore)

We note here that $C(X)$ is *1-convex* if whenever $0 \leq a \leq b \in C(X)$ there exists a $c \in C(X)$ such that $a = bc$. This terminology was introduced by S. Larson in *Convexity conditions on f-rings*, Canad. J. of Math, 38 (1986), 46-64.

(h) *$X$ is an F-space if and only if every ideal of $C(X)$ is convex.* (1956, [GH1])

We say an ideal, I, of $C(X)$ is *convex* if whenever $a \leq c \leq b$ and $a, b \in I$, then $c \in I$.

Many of the above results concern the equivalence of topological properties of $X$ with algebraic conditions on the ring $C(X)$. However, results such as (e) and (f) may be interpreted along a homological vein; i.e. characteristics of $X$ may be related to properties of the category of all $C(X)$-modules. For example,

(e′) *$X$ is basically disconnected if and only if every finitely generated submodule of a projective $C(X)$-module is projective.*

(f′) *$X$ is finite if and only if every submodule of a projective $C(X)$-module is projective.*

Moreover, results such as (g) have directly contributed to the construction of homological theorems; e.g.

(g′) *$X$ is an F-space if and only if the weak dimension of $C(X)$ is no greater than 1.* ([Ve], [Ne], [MW])

(The definition of *weak dimension* will be given in Chapter 2.)

Unfortunately, in spite of the results by [Ve], [Ne], [MW], [Br1], [Br2], and [DM], within the theory of rings of continuous functions, the literature pertaining to homological considerations is relatively limited. Thus, our intention is to survey the terrain, mapping out the possible directions for future research in this area, and identifying any obvious limitations in the application of homological techniques to the study of rings of continuous functions. Implicit in this task lies the danger of confusing, or worse, repelling the reader from this dissertation. However, we believe the possible discoveries of new mathematical relationships, and even deepened insights into relations well known, are well worth the risk.

# CHAPTER 2
## General Preliminaries

### 2.1 Set Theory

Throughout $\kappa$, $\lambda$, $\mu$ will denote cardinals and, when we have the occasion for them, $\alpha$, $\beta$, $\gamma$, and $\delta$ shall denote ordinals. As is customary, we denote $2^\omega$ by $\mathfrak{c}$, and for any cardinal, $\kappa$, we denote the least cardinal greater than $\kappa$ by $\kappa^+$. In order to avoid *sup vs. max* difficulties, and for other reasons which will reveal themselves as this topic develops, we will restrict our considerations to cardinals of the form $\aleph_n$, $n \in \omega$, unless otherwise noted. With respect to axiomatics we shall work within $ZFC$ and make no standing assumptions regarding additional axioms such as $GCH$, i.e., $\aleph_{n+1} = 2^{\aleph_n}$ for every $n \in \mathbb{N}$, although the results in this paper may have greater resonance if the reader assumes $GCH$ throughout.

### 2.2 Topology

All spaces under consideration are assumed to be *Tychonoff*, i.e., for every $x \in X$ and closed set $K \subseteq X$ such that $x \notin K$ there exists an $f \in C(X)$ such that $f(x) = 0$ and $f[K] = 1$. (Note that this is equivalent to saying that $\{\, coz(f)\,\}_{f \in C(X)}$ forms a base for the open sets on $X$, where, for $f \in C(X)$, $coz(f)$=$\{\, x \in X : f(x) \neq 0\,\}$.)

We now briefly review two topological concepts used, if not explicitly, implicitly throughout this presentation; namely, that of a structure space associated with a commutative ring $A$, and the Stone-Čech compactification, $\beta X$, of an arbitrary Tychonoff space $X$. First, consider a commutative ring with identity, $A$, and let $\mathfrak{J}$

be a collection of subobjects of $A$. We note that the collection $\mathfrak{I}$ usually denotes a collection of ideals of $A$. In this case we may construct a compact (although not necessarily Hausdorff) topology on the underlying set $\mathfrak{I}$ by defining a base for the open sets as follows: for $a \in A$ define $U(a)=\{ I \in \mathfrak{I} \mid a \notin I\}$. Then $\{ U(a) \mid a \in A\}$ forms a base for the open sets on the set $\mathfrak{I}$. A topology constructed in this manner on $\mathfrak{I}$ is customarily called the *hull-kernel* topology on $\mathfrak{I}$.

We now supply a sketch of our prefered construction of $\beta X$, for a given Tychonoff space $X$. Although there are numerous methods for constructing $\beta X$, we shall only present the construction of $\beta X$ via the collection of all ultrafilters on $\mathcal{Z}[X]$, where $\mathcal{Z}[X]$ denotes the family of all zero sets of elements of $C(X)$. (Note that a zero set on $X$ is a set of the form $f^{-1}(0)$, for some $f \in C(X)$, i.e., is a complement of a cozero set.) We first remind the reader of the definition of an ultrafilter. Given a lattice, $L$, (i.e. $L$ is a partially ordered set such that for every $a, b \in L$, the supremum and the infimum of $a$ and $b$ exist in $L$) a *filter* on $L$ is a subset $F$ of $L$ which is closed under $\wedge$ (the infimum operation), and if $b \geq a \in F$, then $b \in F$. Then, applying Zorn's Lemma in an appropriate manner one may show that maximal filters exist; we call a maximal filter, $U$, an *ultrafilter* on $L$. Employing this construction of ultrafilters to the lattice $(\mathcal{Z}[X], \cup, \cap)$, one obtains the underlying set upon which the topology of $\beta X$ is constructed. As for a base for the open sets on $\beta X$, we essentially form a hull-kernel topology on the set of all ultrafilters on $\mathcal{Z}[X]$, which is denoted by $Ult(\mathcal{Z}[X])$. Explicitly, for $Z \in \mathcal{Z}[X]$, let $O(Z)=\{ U \in Ult(\mathcal{Z}[X]) \mid Z \notin U\}$. Then $\{ O(Z) \mid Z \in Ult(\mathcal{Z}[X]) \}$ forms a base for the open sets of a compact, Hausdorff topology on $Ult(\mathcal{Z}[X])$; we denote this space by $\beta X$. This notation is justified, as the space constructed in this manner is the unique space (up to homeomorphism) in which $X$ is $C^*$-embedded, i.e., every bounded, continuous function on $X$ has a unique

extension to the space $\beta X$. It is well known ([GJ], Chapter 7) that this property characterizes the Stone-Čech compactification of $X$. (Actually, there are five equivalent conditions which characterize $\beta X$, and a full list will be supplied in Chapter 5.) We note here for the sake of completeness that an equivalent natural development of $\beta X$ may be obtained by applying the construction of the hull-kernel topology (as outlined above) to $Max(C(X))$, the set of all maximal ideals of the ring $C(X)$.

## 2.3 $f$-rings

Our main object of algebraic interest throughout shall be $f$-rings. However, before defining what an $f$-ring is, we shall first define a more general class of objects which subsumes the class of $f$-rings; namely the class of *lattice order groups*. Given a group $(G,+)$, we say that $(G,+)$ is a *lattice ordered group*, or simply, an $l$-group, if there exists a partial order, $\geq$, on $(G,+)$ so that :

(1) $(G,\geq)$ is a lattice;

(2) for $a,b,c \in G$ we have $a \geq b$ implies $a+c \geq b+c$.

From this point it is elementary to extend the above definition to rings so that $(A,+,\cdot,\geq)$ is a *lattice ordered ring*, i.e., $(A,+,\cdot,\geq)$ is a ring which is a lattice ordered group with the additional axiom that the ring multiplication is compatible with the partial order, i.e., $a \leq b$ and $c \geq 0$ imply $ac \leq bc$. As the notation $(A,+,\cdot,\geq)$ is cumbersome, we shall simply say $A$ is a lattice ordered ring. Now, by an *f-ring*, $A$, we mean a lattice-ordered ring such that $a \wedge b = 0$ and $c \geq 0$ implies $a \wedge bc = 0$. Under our assumptions of $ZFC$ this is equivalent to $A$ being isomorphic, as a lattice ordered ring, to a subdirect product of totally ordered rings.

We say an $f$-ring, $A$, satisfies the *bounded inversion* property if for every $u \geq 1$ we have that $u$ is a multiplicative unit. We note here that $C(X)$ satisfies the bounded

inversion property. With regards to subobjects of a given $f$-ring, $A$, there are many classes which are of use to us in this presentation; however, we shall only define two of the most elementary classes here and will develop the rest as needed. First, given a subgroup, $C$, of the $f$-ring $A$, $C$ is said to be *convex* if, whenever $0 \leq a \leq b \in C$, then $a \in C$. If, in addition $C$ is a sublattice of $A$, we shall call $C$ a *convex l-subgroup* of $A$; the collection of convex $l$-subgroups of $A$ will be denoted by $\mathfrak{C}(A)$. Next, given a ring ideal $I$ of $A$, if $I \in \mathfrak{C}(A)$, then we say $I$ is an *l-ideal* of $A$.

## 2.4 Rings and Modules

We will now review some elementary homological concepts. A more rigorous exposition of the required advanced homological results shall be provided in Chapter 3.

Throughout this paper let $A$ be a commutative ring with 1.

(i) Let $M$ be an $A$-module. A *projective resolution of* $M$ is an exact sequence:

$$\Pi : \cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \xrightarrow{d_{-1}} 0$$

where each $P_n$ is a projective $A$-module. We recall for the benefit of the reader that by exactness it is meant that $ker(d_n) = Im(d_{n+1})$ for every $n$. The *length* of a projective resolution $\Pi$ of $M$, denoted $l(\Pi)$, is defined to be $\inf\{\, n+1 \,:\, ker(d_n) \text{ is projective}\}$. We then say that the *projective dimension of* $M$, denoted $pd(M)$, is

$$pd(M) = \inf\{\, l(\Pi) \,:\, \Pi \text{ is a projective resolution of } M \,\}.$$

(ii) For a ring $A$ we define the *global dimension of* $A$ by

$$gD(A) = \sup\{\, pd(M) \,:\, M \text{ an } A\text{-module}\,\}.$$

It is well known ([R], Chapter 9) that:

$$gD(A) = \sup\{\, pd(A/I) \;:\; I \text{ an ideal of } A\,\}\text{,and}$$

$$gD(A) = \sup\{\, pd(I) \;:\; I \text{an ideal of } A\,\} + 1 \;\; if\; gD(A) \neq 0$$

(iii) Given $M$ an $A$-module and replacing "*projective*" with "*flat*" in (1.4)(i) and (1.4)(ii) we obtain the definition of a *flat resolution of* $M$, the *flat dimension of* $M$ (denoted $fd(M)$), and the *weak dimension of* $A$, (denoted $wD(A)$). Explicitly, let $M$ be an $A$-module; a *flat resolution of* $M$ is an exact sequence:

$$\Phi : \cdots \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} X_0 \xrightarrow{d_0} M \xrightarrow{d_{-1}} 0$$

where each $X_n$ is a flat $A$-module. We then define the *length* of a flat resolution $\Phi$ of $M$, denoted by $l(\Phi)$, to be $\inf\{\, n+1 \;:\; ker(d_n) \text{is flat}\,\}$. We then say that the *flat dimension of* $M$, denoted by $fd(M)$, is

$$fd(M) = \inf\{\, l(\Phi) \;:\; \Phi \text{ is a flat resolution of } M\,\}.$$

Analogous to the definition of $gD(A)$, we define the *weak dimension* of $A$ by

$$wD(A) = \sup\{\, fd(M) \;:\; M \text{ an } A\text{-module}\,\}.$$

Similar computational results hold for $wD(A)$ as in (1.4)(ii), as well as

$$wD(A) = sup\{\, wD(A_{\mathfrak{M}}) \;:\; \mathfrak{M} \text{ a maximal ideal of } A\,\}\text{([R], Chapter 9).}$$

(iv) In light of the above definitions it is immediate that:

(a) $gD(A) = 0$ if and only if every $A$-module is projective, i.e., $A$ is *semisimple.* This is equivalent to the condition that $A$ is a finite product of fields.

(b) $gD(A) \leq 1$ if and only if every ideal of $A$ is projective, i.e., $A$ is *hereditary.*

(c) $wD(A) = 0$ if and only if every $A$-module is flat, i.e., $A$ is *von Neumann regular.*

We remind the reader that a ring, $A$, is said to be *von Neumann regular* if, for every $a \in A$, there exists $x \in A$, such that $a^2x = a$.

## 2.5 Categories

Here we remind the reader of a few elementary definitions relative to categories. Although no deep categorical results are required for the results within, the language of category theory rears its head, and thus must be addressed.

We begin by noting that, unless otherwise specifed, all categories considered are abelian, and in fact, no generality would be lost by treating all categories as categories of modules over a fixed ring $A$.(For an explicit definition of an abelian category, the reader is reffered to [R], Chapter 2.) In light of these comments, we may simplify the definitions of right and left exact functors. Recall that, given two categories $\mathfrak{A}$ and $\mathfrak{B}$, a *functor* $F : \mathfrak{A} \longrightarrow \mathfrak{B}$, is a function such that:

(a) for each object $M \in \mathfrak{A}$ we have $F(M) \in \mathfrak{B}$

(b) for each morphism $f \in \mathfrak{A}$, we have $F(f)$ is a morphism in $\mathfrak{B}$.

(c) for $f, g$, morphisms in $\mathfrak{A}$, we have $F(fg) = F(f)F(g)$ if $F$ is covaraint, and $F(fg) = F(g)F(f)$ if $F$ is contravariant.

(d) for $M \in \mathfrak{A}$ and $1_M$ the identity morphism in $\mathfrak{A}$, we have $F(1_M) = 1_{F(M)}$.

Now, let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence in $\mathfrak{A}$. Then, we call a functor, $F$, *right exact* if

$$F(L) \longrightarrow F(M) \longrightarrow F(N) \longrightarrow 0$$

is an exact sequence in $\mathfrak{B}$, and we say $F$ is *left exact* if

$$0 \longrightarrow F(L) \longrightarrow F(M) \longrightarrow F(N)$$

is an exact sequence in $\mathfrak{B}$. If it is the case that $F$ is both right and left exact, we then say that $F$ is an *exact functor*.

# CHAPTER 3
## Homological Preliminaries and Change of Rings

The purpose of this chapter is to review the more advanced homological concepts needed for the development of this disertation. In the first section we will review results by M. Auslander and Osofsky which, in some ways, are the genesis for the author's thesis on global dimensions of rings of continuous functions. These results are indeed striking when set against the backdrop of the more traditional work done on global dimensions of rings satisfying some sort of finiteness conditions, e.g., Noetherian rings. In an attempt to provide the reader with a sense of the set theoretic motivations for the work which follows, in many cases, sketches of proofs, or at least an exposition of the core constructions involved in the proofs, are supplied. The second and final section of this chapter contains a result, essential to the proof of the main theorem in Chapter 4, concerning the flat dimension of a module under a change of rings. An analogous result for the projective dimension of a module under a change of rings is readily found in the literature, and its proof is relatively straightfoward. Unfortunately, the author was unable to locate the needed result regarding flat dimension and change of rings; in fact, it seems as if this result is not commonly known amongst practictioners of homological algebra. To make matters worse, the standard proof of the analogous result for projective dimension under a change of rings does not generalize to the case for flat dimension. However, this problem does give way under the techniques of spectral sequences. Thus, the proof offered in Section 2 of this crucial result is achieved by such machinery. To those unfamiliar with the language of spectral sequences, we apologize in advance for not including a development of the

subject within; for such a task would require *many* pages, take us far afield, and is inappropriate in this setting. This being said, a few basic definitions are supplied so that one might at least be able to read the statement of the theorem required for Chapter 4. In addition, we strongly suggest Chapter 11 of [R] as a guide.

## 3.1 Advanced Homological Concepts

We begin this section with two elementary lemmas concerning projective dimension. The follwing may be found as exercises in [R], Chapter 9.

**Lemma 3.1.1** *Let* $\{ M_\lambda \}_{\lambda<\gamma}$ *be a family of A-modules. Then*

$$pd(\oplus_{\lambda<\gamma}(M_\lambda)) = \sup_{\lambda<\gamma} pd(M_\lambda)$$

*and*

$$fd(\oplus_{\lambda<\gamma}(M_\lambda)) = \sup_{\lambda<\gamma} fd(M_\lambda).$$

**Lemma 3.1.2** *Let* $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ *be an exact sequence of A-modules such that M is projective. Then either all three modules are projective, or else* $pd(L) = pd(N) + 1$.

The following is due to Auslander and dates back to 1956. Arguably, this is the theorem which launches the theory of homological algebra pertaining to rings without finitistic conditions.

**Theorem 3.1.3** (Auslander's Lemma) *Let* $M$ *and* $\{M_\lambda\}_{\lambda<\gamma}$ *be A-modules where* $\gamma$ *is an ordinal and* $M = \cup_{\lambda<\gamma} M_\lambda$. *Assume for every* $\alpha < \beta < \gamma$ *we have* $M_\alpha \subseteq M_\beta$, *and for every* $\delta < \gamma$, $pd(M_\delta / \cup_{\beta<\delta} M_\beta) \leq k \in \omega$. *Then* $pd(M) \leq k$.

**Proof.** The proof of the above proceeds by induction on $k$. Indeed, if $k = 0$, then for every $\delta < \gamma$ we have $pd(M_\delta / \cup_{\beta<\delta} M_\beta)=0$, i.e., $M_\delta / \cup_{\beta<\delta} M_\beta$ is projective. Thus,

$M_\delta \cong (\cup_{\beta<\delta} M_\beta) \oplus (M_\delta / \cup_{\beta<\delta} M_\beta)$ and so $M \cong \oplus_{\delta<\gamma}(M_\delta / \cup_{\beta<\delta} M_\beta)$. It follows from 3.1.1 above that $pd(M)$=0. Assuming the result holds for all $j < k$, let $M = \cup_{\delta<\gamma} M_\delta$. Using the existence of short exact sequences

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0,$$

and for every $\delta < \gamma$,

$$0 \longrightarrow K_\delta \longrightarrow F_\delta \longrightarrow M_\delta \longrightarrow 0$$

and 3.1.2 above, it can be shown that $K = \cup_{\delta<\gamma} K'_\delta$, where $\{ K'_\delta \}_{\delta<\gamma}$ form a chain under inclusion, and for every $\delta < \gamma$, $pd(K'_\delta / \cup_{\beta<\delta} K'_\beta) \leq k-1$. From the induction hypothesis we conclude that $pd(K) \leq k-1$. Invoking 3.1.2 again, we obtain the desired result. □

The next two results by Osofsky provide upper and lower bounds respectively for $gD(A)$ in terms of cardinalities for generating sets of ideals of $A$.

**Theorem 3.1.4** *Let $A$ be a ring so that every ideal of $A$ is generated by at most $\aleph_n$ elements. Then*

$$gD(A) \leq wD(A) + n + 1. \ [Os1]$$

This result is proved by invoking the following two results from [Os1], stated here without further comment:

**Proposition 3.1.5** *If $M$ is an $\aleph_n$-related flat module (i.e. $M$ is flat and there exists a short exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ where $F$ is free and $K$ is $\aleph_n$-generated) then $pd(M) \leq n+1$.*

**Proposition 3.1.6** *Let $A$ be a ring such that every ideal of $A$ is generated by at most $\aleph_n$ elements. Then any submodule of a free $A$-module on a set of at most $\aleph_n$ elements is generated by at most $\aleph_n$ elements.*

To employ 3.1.5 and 3.1.6 above, observe if $wD(A) = \infty$ then the result in 3.1.4 is trivial. Thus, assume $wD(A) < \infty$, let $A/I$ be a cyclic $A$-module, and consider a projective resolution $\Pi(A/I)$ of $A/I$. By 3.1.6 every $P_i$ in the projective resolution $\Pi(A/I)$ is free on $\aleph_n$ generators. If $d_k(P_k)$ is flat then 3.1.5 implies $pd(P_k) \leq n+1$, and hence $pd(A/I) \leq k+n+1$. Recalling 1.4(ii) we obtain

$$gD(A) \leq sup\{\, fd(A/I) + n + 1 \,:\, I \text{ an ideal of } A\} = wD(A) + n + 1.$$

We note that in the above, 3.1.5 is proved essentially using a version of Auslander's Lemma, and 3.1.6 is proved by a standard induction on $n$.

In order to state the second theorem of Osofsky, which will provide lower bounds for $gD(A)$, we first introduce a definition:

**Definition 3.1.7** *Let $\{e_\gamma\}_{\gamma<\alpha}$ be a collection of idempotents where $\alpha$ is an ordinal. We say the collection $\{e_\gamma\}_{\gamma<\alpha}$ is independent if*

$$(\Pi_{k\leq n} e_{\gamma_k}) \cdot (\Pi_{j\leq m}(1 - e_{\delta_j})) \neq 0,$$

*whenever $\{e_{\gamma_k}\}_{k\leq n} \cap \{e_{\delta_j}\}_{j\leq m} = \emptyset$, where $m, n < \omega$.*

**Theorem 3.1.8** *Let $\{e_\gamma\}_{\gamma<\alpha}$ be a collection of independent idempotents of $A$ and let $I = \Sigma_{\gamma<\alpha} e_\gamma A$. If there exists an $n \in \omega$ such that no ordinal of cardinality $< \aleph_n$ is cofinal in $\alpha$ then $pd(I) \geq n$.* [Os1]

The proof of the above proceeds via a direct construction of a rather particular projective resolution of $I$, of which we now give a brief account. In order to construct the appropriate projective resolution all that must be done is to specify the modules, $P_n$, in the sequence, and the maps, $d_n$, connecting them. To this end we define:

**Definition 3.1.9**

*(a)* $P_n(\mathcal{E}) = \bigoplus_{i_0 < i_1 < \ldots < i_n}(A < i_o, ..., i_n >)$, *such that*

$$< i_0, ..., i_n >: (\alpha)^{n+1} \longrightarrow A$$

*is a function which assumes the value 0 everywhere except at* $(i_0, ..., i_n)$, *where it takes on the value* $\prod_{0 \le j \le n+1} e(\gamma_j)$, *and* $(i_o, ..., i_n)$ *varies over all subsets of* $\mathcal{E}$ *of size* $n+1$ *which satisfy the given ascending order condition.*

*(b)* $d_n : P_n(\mathcal{E}) \longrightarrow P_{n-1}(\mathcal{E})$ *by* $d_0(< i_0 >) = e(i_0)$ *and*

$$d_n(< i_0, ..., i_n >) = \sum_{j=0}^{n}[(-1)^j < i_0, ...\hat{i}_j, ..., i_n > (e(\gamma_j))]$$

*where* $< i_o, ..., \hat{i}_j, ..., i_n >$ *is* $< i_0, ..., i_n >$ *with* $i_j$ *deleted.*

With the above definitions in place it can be shown that

$$\mathfrak{P}(\mathcal{E}) : \cdots \longrightarrow P_n(\mathcal{E}) \xrightarrow{d_n} P_{n-1}(\mathcal{E}) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0(\mathcal{E}) \xrightarrow{d_0} I \longrightarrow 0$$

is a projective resolution of $I$.

The result cited in this section has a two-fold importance in this presentation. First, a direct application of it to localizations of $C(X)$ will assist in providing a fundemental result concerning $P$-spaces which will be presented in Chapter 7. Second, its proof employs a certain projective resolution which serves as the inspiration for the projective resolution used to prove the main result in Chapter 7. It is for these reasons that we not only state the result here, but also sketch the construction of the above mentioned resolution.

We first recall a familiar definition from ring theory:

**Definition 3.1.10** *We say a ring, $A$, is a valuation domain if $A$ is a domain, and if for every pair $a, b \in A$, we have either that $a$ is a multiple of $b$, or $b$ is a multiple of $a$.*

**Remark 3.1.11** We note that the above definition is equivalent to the condition that the ideals of $A$ form a chain under inclusion.

**Theorem 3.1.12** ([Os1]) *Let $A$ be a valuation domain and $n \in \omega$. For an ideal, $I$, of $A$ we have $pd(I) = n+1$ if and only if the ordinal $\omega_n$ is cofinal in the chain of principal ideals $\{(a) : a \in I\}$.*

As stated above we will now provide a sketch of the projective resolution employed in the proof of the above. The resolution is that of an ideal, $I$, of a given valuation domain, $A$, such that $I$ is generated by a chain of idempotents, say $\mathfrak{E} = \{e_\lambda\}_{\lambda<\gamma}$. In order to construct the necessary resolution we must first state some definitions.

**Definition 3.1.13**

*(a)* $P_{-1}(\mathfrak{E}) = I$, *where $I$ is generated by the chain $\mathfrak{E}$*

*(b)* *For $n \geq 0$ let*

$$P_n(\mathfrak{E}) = \bigoplus_{e_0 > \ldots > e_n} (A < e_0, \ldots, e_n >)$$

*where $\{e_0 > \ldots > e_n\}$ ranges over all subsets of $\mathfrak{E}$ of size $n+1$ satisfying the given decreasing order condition. Note then that $P_n(\mathfrak{E})$ is actually a free $A$-module.*

*(c)* *For $n \geq 1$ define $d_n : P_n(\mathfrak{E}) \longrightarrow P_{n-1}(\mathfrak{E})$ by $d_n(< e_0, \ldots, e_n >) =$*

$$[\sum_{i=0}^{n-1} (-1)^i < e_0, \ldots, \hat{e}_i, \ldots, e_n >] + (-1)^n e_n e_{n-1}^{-1} < e_0, \ldots, e_{n-1} >$$

*where $< e_0, \ldots, \hat{e}_i, \ldots, e_n >$ is $< e_0, \ldots, e_n >$ with $e_i$ deleted, and $e_n^{-1}$ is a type of "inverse" for $e_n$.*

*(d)* *For $n = 0$ define $d_0 : P_0(\mathfrak{E}) \longrightarrow P_{-1}(\mathfrak{E})$ by $d_0(ae_0) = ae_0$*

(e) *Given* $e \in \mathfrak{E}$ *and* $n \geq 0$, *let* $e^* : P_n(s(e)) \longrightarrow P_{n+1}(\bar{s}(e))$ *be defined by* $e^*(< e_0, ..., e_n >) =< e, e_0, ..., e_n >$, *where* $s(e) = \{f \in \mathfrak{E} : f < e\}$ *and* $\bar{s}(e) = \{f \in \mathfrak{E} : f \leq e\}$.

(f) *Given* $e \in \mathfrak{E}$ *and* $n = -1$ *define* $e^* : P_{-1}(\bar{s}(e)) \longrightarrow P_0(\bar{s}(e))$ *by* $e^*(ae) = ae$.

With the above definitions one may show that

$$\mathcal{P}(\mathfrak{E}) : \cdots \longrightarrow P_n(\mathfrak{E}) \xrightarrow{d_n} P_{n-1}(\mathfrak{E}) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0(\mathfrak{E}) \xrightarrow{d_0} I \longrightarrow 0$$

is a projective resolution of $I$.

To conclude this section we recall a result which characterizes rings, $A$, for which $wD(A) \leq 1$, in terms of their localizations at maximal ideals. The result may be found in [Gl], and is as follows:

**Theorem 3.1.14** *Let $A$ be a ring. Then $wD(A) \leq 1$ if and only if $A_\mathfrak{m}$, the localization of $A$ at $\mathfrak{m}$, is a valuation domain for every $\mathfrak{m} \in Max(A)$.*

## 3.2 Change of Rings

The purpose of this section is to prove a result which is pivotal in the proof of the main theorem of Chapter 4. It concerns the relationship between the flat dimension of a module over two distinct rings, where the change of rings is given by restriction of scalars, i.e., by a ring homomorphism between the two rings under consideration. The proof of the theorem in question uses the language of spectral sequences, a language with which some readers of this dissertation might not be familiar. A few of the most elementary definitions regarding this subject, e.g., complexes and the $Tor_n(\cdot, M)$ functor, are supplied, but in no way are the definitions offered

within adequate preparation for the uninitiated to be able to fully understand the theorems within. Indeed, this is an understatement. With that in mind, those who are unfamiliar with spectral sequences and do not wish to take the results in this section on faith are referred to Chapter 11 of [R]. If, however, the reader is familiar with the techniques of spectral sequences and their power, the final result in this section should come as no surprise, it is a simple analogue of a well known result concering projective dimension under restriction of scalars. Throughout, $R$ and $T$ denote commutative rings with identity, and $\mathfrak{M}_R$, $\mathfrak{M}_T$, the categories of $R$ and $T$ modules, respectively. We note that all which is contained within this section, save the final result, may be found in [R], Chapters 6, 9, and 11.

**Definition 3.2.1** *A complex is a sequence* $\{\, M_n,\, d_n \,\}_{n\in\mathbb{Z}}$*, such that, for every* $n \in \mathbb{Z}$*,* $M_n \in \mathfrak{M}_R$*,* $d_n : M_n \longrightarrow M_{n-1}$*, and* $d_{n-1}d_n = 0$*.*

**Definition 3.2.2** *Let* $C_1 = \{\, M_n,\, d_n \,\}$ *and* $C_2 = \{\, N_n,\, h_n \,\}$ *be complexes. A chain map* $f : C_1 \longrightarrow C_2$ *is a sequence* $\{\, f_n : M_n \longrightarrow N_n \,\}$ *of R-module homomorphisms such that, for every* $n \in \mathbb{Z}$*,* $f_{n-1}d_n = h_n f_n$*.*

**Remark 3.2.3** Given a fixed ring, $R$, a category, $\mathcal{C}$, may be defined whose objects are all complexes of $R$-modules, and whose morphisms are all chain maps between such complexes. For the detials, check [R], Chapter 2.

**Definition 3.2.4** *Let* $C = \{\, M_n,\, d_n \,\}$ *be a complex. Define the* $n^{th}$ *homology module of* $C$*, denoted* $H_n(C)$*, by* $H_n(C) = ker(d_n)/Im(d_{n+1})$*.*

**Remark 3.2.5** Note first that the above definition makes sense as $C$ is a complex, hence $Im(d_{n+1}) \subseteq ker(d_n)$. Second, if $C_1$ and $C_2$ are complexes, and $f : C_1 \longrightarrow C_2$ is a chain map, then one can show if $n \in \mathbb{Z}$, then there exists a module homomorphism

$H_n(f) : H_n(C_1) \longrightarrow H_n(C_2)$. Indeed, it can be proved that, for $\mathcal{C}$, the category of all complexes over $R$, for every $n \in \mathbb{Z}$, $H_n : \mathcal{C} \longrightarrow \mathfrak{M}_R$ defines a functor.

**Definition 3.2.6** *Consider a complex, $C$, of $R$-modules of the form*

$$P : \cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} 0.$$

*Now, let $B$ be a fixed $R$-module. Upon tensoring the complex $C$ over $R$ with the module $B$, we obtain*

$$P \otimes B : \cdots \longrightarrow P_n \otimes B \xrightarrow{d_n \otimes 1_B} P_{n-1} \otimes B \xrightarrow{d_{n-1} \otimes 1_B} \cdots \xrightarrow{d_1 \otimes 1_B} P_0 \otimes B \xrightarrow{d_0 \otimes 1_B} 0.$$

*Next, recall that any $R$-module, $M$, has a projective resolution, say $P_M$, which, upon deleting $M$ from the front of this resolution, yields a complex of the form $P$. Then, for every $n \in \mathbb{N}$, we define the $R$-module, $Tor_n(M, B) = H_n(P \otimes B)$, where $P$ is obtained from a projective resolution of $M$, by deleting $M$ from the front of the resolution.*

As our primary concern here is the flat dimension of modules, the following two theorems are offered. The first relates the $Tor_n(\cdot, B)$ functor to the flat dimension of a module. The second is essentially a special case of the first, and concerns rings with weak dimension less than or equal to 1.

**Theorem 3.2.7** *Let $M$ be an $R$-module. Then the following are equivalent:*

*(a)* $fd(M) \leq n$;

*(b)* $Tor_k(M, B) = 0$ *for all* $k \geq n + 1$, *and for all $R$-modules $B$;*

*(c)* $Tor_{n+1}(M, B) = 0$ *for all $R$-modules $B$.*

**Theorem 3.2.8** *For a ring $R$, the following are equivalent:*

*(a) every ideal of $R$ is flat;*

*(b)* $Tor_2(M, B) = 0$ *for all R-modules A,B;*

*(c)* $wD(R) \leq 1$.

Before the statement and proof of the final result, a few definitions are offered in hope that they might at least allow those unfamiliar with spectral sequences to be able to read the theorem and corollary which follow.

**Definition 3.2.9** *A bigraded module is a collection of R-modules of the form* $\{ M_{(p,q)} \}_{(p,q)\in(\mathbb{Z},\mathbb{Z})}$.

**Definition 3.2.10** *A spectral sequence is a sequence* $\{ E^n,\, d^n \}_{n\geq 1}$ *of bigraded modules and maps such that, for every* $n$, $d^{n-1}d^n = 0$, *and* $E^{n+1} = H(E^n, d^n)$ *as bigraded modules.*

**Remark 3.2.11** The above concept could be taken in a *loose* manner to be a three dimensional generalization of the concept of a complex of modules, with $H(E^n, d^n)$ defined appropriately.

**Remark 3.2.12** Let $R$ and $T$ be rings and $\phi : R \longrightarrow T$ be a ring homomorphism. Given $M \in \mathfrak{M}_T$, we may define an action of $R$ on $M$ by $r \cdot m = \phi(r)m$, where $r \in R$, $m \in M$, and $\phi(r)m$ is the action of $\phi(r) \in T$ on $m \in M$ as a $T$-module. Thus, we may consider $M$ as an $R$-module. The question which now concerns us involves the relationship between the flat dimension of $M$ over $T$, denoted $fd_T(M)$, and the flat dimension of $M$ over $R$, denoted $fd_R(M)$.

The theorem which follows is a simplfied version of a theorem due to Grothendieck. Again, its statement and proof may be found in [R], Chapter 11. The notation "$\Longrightarrow_p$" is used to indicate a form of convergence not unlike the approximation of a group by a series of factor modules, a concept the reader may be more familiar with.

**Theorem 3.2.13** *Let $R$ and $T$ be rings, and $\phi : r \longrightarrow T$ a ring homomorphism. Given $N \in \mathfrak{M}_R$ and $M \in \mathfrak{M}_T$, there exists a spectral sequence*

$$Tor_p^T(Tor_q^R(N,T), M) \Longrightarrow_p Tor_{p+q}^R(N,M).$$

Finally, we arrive at the desired result.

**Corollary 3.2.14** *Let $\phi : R \longrightarrow T$ be a ring homomorphism and $M \in \mathfrak{M}_T$. Then $fd_R(M) \leq fd_T(M) + fd_R(T)$.*

**Proof:** Assume $fd_T(M), fd_R(T) < \infty$; say, $fd_T(M) = j$ and $fd_R(T) = k$. Then, by 3.2.13, there exists a spectral sequence

$$Tor_p^T(Tor_q^R(N,T), M) \Longrightarrow_p Tor_{p+q}^R(N,M),$$

for every $N \in \mathfrak{M}_R$. Since $fd_T(M) = j$, we have

$$Tor_{p'}^T(Tor_q^R(N,T), M) = 0, \text{ for } p' > j.$$

Similarly, if $q' > k$, then $Tor_{q'}^R(N,T) = 0$. Thus, for $n = p + q > j + k$, as either $p > j$ or $q > k$, we have

$$Tor_p^T(Tor_q^R(N,T), M) = 0.$$

Recalling that

$$Tor_p^T(Tor_q^R(N,T), M) \Longrightarrow_p Tor_{p+q}^R(N,M),$$

it follows that $Tor_n^R(N,M) = 0$, for $n > p + q$. As $N \in \mathfrak{M}_R$ was arbitrary, invoking 3.2.7, we conclude that

$$fd_R(M) \leq j + k = fd_T(M) + fd_R(T),$$

and the result is complete. □

# CHAPTER 4
## Weak Dimension: $F$-Spaces and $SV$-Spaces

### 4.1 F-Spaces

In this section we address the problem of identifying the flat ideals of a particular class of $f$-rings via the order theoretic properties of these rings. Although this problem is interesting in its own right, here it is treated as a means to an end; namely, the completion of the proof of the central result of this chapter concerning an upper bound for the weak dimension of an $SV$-ring, a notion which will be defined in the second section of this chapter. To motivate, we begin with a review of $F$-spaces.

The notion of an $F$-space first appeared in the literature in 1956 in a paper by Gillman and Henriksen entitled *Concerning rings of continuous functions in which every finitely generated ideal is principal.* Indeed, the title of this paper provides the original definition of an $F$-space. Precisely:

**Definition 4.1.1** *A Tychonoff space, $X$, is said to be an $F$-space if $C(X)$ is Bézout, i.e., every finitely generated ideal of $C(X)$ is principal.*

It may be shown that the above ideal theoretic condition is equivalent to the topological condition that every cozero set of $X$ is $C^*$-embedded in $X$. Utilizing this topological condition in conjunction with the well known fact ([GJ],Chapter 14) that $X$ is an $F$-space if and only if $\beta X$ is an $F$-space, one may show that the space $\beta\mathbb{N}$, the Stone-Čech compactification of the natural numbers, is an $F$-space. Upon further investigation of this condition on cozero sets, one might be inclined to believe that $F$-spaces have a high degree of disconnectivity (a notion which we will make precise

in Chapter 5). However, it is well known that, for $\mathbb{R}^+ = [0, \infty)$, $(\beta\mathbb{R}^+) - \mathbb{R}^+$ is, in fact, a connected $F$-space.([GJ], Chapters 6,14)

At this point a theorem is offered, which, as a special case, characterizes $C(X)$ when $X$ is an $F$-space, in terms of the ring and order theoretic properties of $C(X)$, as well as in terms of its weak dimension. This result is stated, in general, for semiprime $f$-rings with bounded inversion, a class which contains the class of all rings of the form $C(X)$, which we will from now on denote by $\mathcal{C}(\mathfrak{X})$. As usual, we must first supply some definitions.

**Definition 4.1.2**

*(a) A ring, $A$, is called semiprime if it has no nonzero nilpotent elements, i.e., if $a \in A$ and there exists an $n \in \mathbb{N}$ such that $a^n = 0$, then $a = 0$.*

*(b) Given an ideal, $I$, of $A$, we say $I$ is semiprime if the factor ring $A/I$ is a semiprime ring. The reader may easily verify that this condition is equivalent to $a^n \in I$, for some $n \in \mathbb{N}$, implies $a \in I$.*

*(c) An $f$-ring, $A$, is said to satisfy the $n^{th}$-convexity condition if, whenever $0 \leq a, b \in A$ such that $a \leq b^n$, then there exists $c \in A$ such that $a = bc$.*

*(d) A ring, $A$, is said to be square root closed if for every $0 \leq a \in A$ there exists $b \in A$ such that $a = b^2$.*

**Remark 4.1.3** We note that in [L] it was shown that if an $f$-ring, $A$, satisfies the $n^{th}$-convexity condition for $n \geq 1$, then every semiprime ideal of $A$ is an $l$-ideal.

We are now ready to state the promised result. It is due to Martinez and Woodward and may be found in [MW]:

**Theorem 4.1.4** *Let $A$ be a semiprime $f$-ring with bounded inversion. Then the following are equivalent:*

*(a) $A$ is Bézout;*

*(b) $A$ is 1-convex;*

*(c) every ideal of $A$ is convex;*

*(d) every ideal of $A$ is an $l$-ideal;*

*(e) for $0 \le a, b \in A$ we have $(a,b) = (a+b)$, where $(a,b)$ is the ideal generated by $a$ and $b$;*

*(f) for every $\mathfrak{m} \in Max(A)$, $A_{\mathfrak{m}}$ is a valuation domain.*

**Remark 4.1.5** We note that in light of the result stated in 3.1.14, the above conditions are equivalent to $wD(A) \le 1$.

Before proceeding to the theorem on flat ideals, it is necessary to review an alternate definition of flatness for modules over a given ring, $A$. Although this definition is equivalent to the more common definition of flatness, it is more obscure, and thus we remind the reader of it now.

**Definition 4.1.6** *Let $M$ be an $A$-module. Then $M$ is said to be a flat $A$-module, if the following condition holds:*

> Given $a_i \in A$ and $m_i \in M$, for $1 \le i \le n$, if $\sum_{i=1}^{n} a_i m_i = 0$, then there exists a matrix $(b_{ij})$, a vector $(x_j)$, and a $k \in \mathbb{N}$, so that $b_{ij} \in A$, $x_j \in M$, $1 \le i \le n$, $1 \le j \le k$, where $\sum_{j=1}^{k} b_{ij} x_j = m_i$ for each $i$, and $\sum_{i=1}^{n} a_i b_{ij} = 0$ for each $j$.

Recalling that $wD(A) = \sup\{ fd(I) \,:\, I \text{ an ideal of } A \} + 1$, if $wD(A) \neq 0$, we see that 4.1.4 implies that, for an $F$-space, every ideal of $C(X)$ is flat. This result was proved by [Ne], and by [Ve], independently. It also raises the question of when an ideal of an arbitrary $C(X)$ is flat. Indeed, in the same paper where Večtomov proves every ideal of a Bézout $C(X)$ is flat, he also shows the following:

**Theorem 4.1.7** [Ve]: *Let $I$ be a semiprime ideal of $C(X)$. Then $I$ is a flat $C(X)$-module.*

Večtomov's proof of the above relies heavily on the underlying topological space $X$, in the sense that the construction of the appropriate matrix $(b_{ij})$ and vector $(x_j)$ in 4.1.6 utilizes filters on the space $X$. However, if one abandons the space $X$ and considers the order theoretic properties of the $f$-ring $C(X)$, then the result is seen to be true for a lager class of $f$-rings which contains the class $\mathcal{C}(\mathfrak{X})$. Precisely:

**Theorem 4.1.8** *Let $A$ be a 2-convex, square root closed, semiprime, $f$-ring with bounded inversion. Then every semiprime ideal of $A$ is a flat $A$-module.*

**Proof:** First, note that if $A$ is a 2-convex $f$-ring, then, for $a, b \in A$ with $|a| \leq b^2$, there exists $c \in A$ such that $a = bc$. In fact, if $a \in A$, we have $a = a^+ - a^-$, where $a^+ = a \vee 0$ and $a^- = (-a) \vee 0$. Thus, as $a^+, a^- \leq |a|$, it is the case that $0 \leq a^+, a^- \leq b^2$, and so, by 2-convexity, there exists $c_1, c_2 \in A$ such that $a^+ = bc_1$ and $a^- = bc_2$. It follows that $a = a^+ - a^- = bc_1 - bc_2 = b(c_1 - c_2)$, where $c_1 - c_2 \in A$. Now, let $I$ be a semiprime ideal of $A$. By 4.1.4, we have that $I$ is an $l$-ideal since $A$ is 2-convex. We will employ 4.1.6 to prove that $I$ is a flat $A$-module. To this end, let $m_i \in I$ and $a_i \in A$, $1 \leq i \leq n$, where $\sum_{i=1}^{n} a_i m_i = 0$. As $I$ is an $l$-ideal, $\vee_{i=1}^{n} |m_i| \in I$. Since $A$ is square root closed there exists $d \in A$ such that $d^2 = \vee_{i=1}^{n} |m_i| \in I$, hence

$d \in I$, as $I$ is semiprime. Note that, for every $i$, we have $|m_i| \leq d^2$, and so, for every $i$ there exists $r_i \in A$ with $m_i = r_i d$. As

$$0 = \sum_{i=1}^{n} a_i m_i = \sum_{i=1}^{n} a_i(r_i d) = (\sum_{i=1}^{n} a_i r_i)d,$$

it is the case that $(\sum_{i=1}^{n} a_i r_i)^2 d = 0$. Using the fact that $A$ is square root closed again, we obtain $f \in A$ such that $f^2 = d$. Thus $((\sum_{i=1}^{n} a_i r_i)f)^2 = 0$; consequently, by the fact that $A$ is semiprime, $(\sum_{i=1}^{n} a_i r_i)f = 0$. Defining $b_{i1} = r_i f$, and $x_1 = f$, we obtain $\sum_{i=1}^{n} a_i b_{i1} = \sum_{i=1}^{n} a_i(r_i f) = 0$ and $x_1 b_{i1} = r_i f^2 = r_i d = m_i$, where $b_{i1} \in A$, and $x_1 \in I$, by the semiprimeness of $I$, for every $i$. Therefore, by 4.1.6, $I$ is a flat $A$-module. □

## 4.2 SV-Spaces

We now turn our attention to the problem of determining an upper bound for a particular class of $f$-rings; namely the class of *SV-rings with finite rank.* This class shall be defined later in this section, but for now it suffices to say that this class contains the members of $\mathcal{C}(\mathfrak{X})$ which are Bézout.

This section begins with motivation for the definition of an $SV$-ring, and then provides a further review of basic concepts from the theory of lattice ordered groups. It concludes with the statement and proof of a result providing upper bounds for the weak dimension of $f$-rings which belong to the class alluded to above.

Let $A$ be an $f$-ring, $\mathfrak{m} \in Max(A)$, and define

$$\mathcal{O}(\mathfrak{m}) = \{\, a \in A : \text{there exists}\, b \notin \mathfrak{m}\, \text{such that}\, ab = 0 \,\}.$$

Now, we wish to consider classes of $f$-rings with bounded inversion, $\mathfrak{C}$, such that $A \in \mathfrak{C}$ if and only if $A/\mathcal{O}(\mathfrak{m}) \in \mathfrak{C}$ for every $\mathfrak{m} \in A$. In addition, for each such class $\mathfrak{C}$, we wish to consider a class of Tychonoff spaces, $\mathfrak{T}$, such that $X \in \mathfrak{T}$ if and only if

$C(X) \in \mathfrak{C}$. It can be shown that the class of $F$-spaces is such a class. That this is indeed true, may be seen by taking $\mathfrak{C}$ to be the class of 1-convex $f$-rings with bounded inversion. Then, as an $l$-homomorphic image of a 1-convex $f$-ring is again 1-convex, by 4.1.4, it is immediate that $X \in \mathfrak{T}$ if and only if $C(X) \in \mathfrak{C}$. In addition, it will become evident after it has been defined, that the class of all $SV$-spaces is also such a class of Tychonoff spaces, which, in fact, properly contains the class of all $F$-spaces.

The development of $SV$-spaces and rings begins with some definitions:

**Definition 4.2.1** *A totally ordered field, $\mathbb{F}$, is said to be real closed, if every positive element has a square root, and every monic polynomial of odd degree in one variable over $\mathbb{F}$ has a root in $\mathbb{F}$.*

**Definition 4.2.2** *An integral domain, $D$, is called real closed if:*

*(a) $D$ is totally ordered;*

*(b) every positive element of $D$ has a square root;*

*(c) every monic polynomial in one variable of odd degree over $D$ has a root in $D$;*

*(d) for $0 < a < b \in D$ there exists $c \in D$ such that $ac = b$.*

With regard to the above definition, Cherlin and Dickman prove in [CD] that, for every $P \in Spec(C(X))$, (where, as is customary, $Spec(C(X))$ denotes the collection of all prime ideals of $C(X)$) $C(X)/P$ is a real closed ring if and only if $C(X)/P$ is a valuation domain. Then, defining $P \in Spec(C(X))$ to be *real closed* if and only if $C(X)/P$ is a real closed ring, we arrive at the following definition:

**Definition 4.2.3** *Call $C(X)$ a survaluation ring, if every $P \in Spec(C(X))$ is real closed.*

Extending this definition to $f$-rings in general, we have:

**Definition 4.2.4** *Let $A$ be an $f$-ring. If, for every $P \in Spec(A)$, $A/P$ is a valuation domain, we call $A$ a survaluation ring, or, for short, an $SV$-ring.*

**Remark 4.2.5** Note that by 4.1.4, every Bézout $f$-ring is in fact an $SV$-ring.

This quickly leads us to the definition of an $SV$-space.

**Definition 4.2.6** *Call a Tychonoff space, $X$, an $SV$-space, if $C(X)$ is an $SV$-ring.*

Before moving on to the definition of $f$-rings with finite rank, it is first necessary to review some basic concepts from the theory of lattice ordered groups. Once again, terminology must be fixed.

**Definition 4.2.7**

(a) *Let $A$ be an $f$-ring and $I$ an $l$-ideal of $A$. Then $I$ is said to be a prime $l$-ideal if, whenever $a, b \in A$ such that $ab \in I$, then either $a \in I$ or $b \in I$.*

(b) *Let $A$ be an $f$-ring and $C \in \mathfrak{C}(A)$, i.e. $C$ is a convex $l$-subgroup of $A$. We say $C$ is an $l$-prime convex $l$-subgroup of $A$ if, whenever $a, b \in A$ such that $a \wedge b = 0$, either $a \in C$ or $b \in C$.*

(c) *Denote the collection of minimal prime $l$-ideals of $A$ by $Min(A)$, and the collection minimal $l$-prime $l$-ideals of $A$ by $Min_l(A)$.*

(d) *Let $G$ be a lattice ordered group and $S \subseteq G$. Then denote by $S^\perp = \{h \in G : |h| \wedge |g| = 0\, for\ all\, g \in G\}$. If $S = \{g\}$, write $g^\perp = \{g\}^\perp$.*

(e) *Let $G$ be a lattice ordered group and $P \in \mathfrak{C}(G)$. Then $P$ is said to be a polar of $G$ if $(P^\perp)^\perp = P$. Now denote the collection of all polars of $G$ by $\mathcal{P}(G)$. If,*

*in addition, $P$ is of the form $(g^{\perp})^{\perp}$, then we say that $P$ is a principal polar; denote the collection of principal polars of $G$ by $\mathcal{P}_r(G)$.*

**Remark 4.2.8** We note here that the definition given in 4.2.7(b), for $P \in \mathfrak{C}(G)$, an $l$-prime convex $l$-subgroup, is equivalent to the condition that the factor $l$-group, $G/P$, is totally ordered under the order inherited from $G$. Also, regarding $l$-prime, convex $l$-subgroups, if $A$ is a semiprime $f$-ring, then $Min(A) = Min_l(A)$. With respect to the polars of $G$, if $A$ is again a semiprime $f$-ring, then every polar of $A$ is an $l$-ideal. (Proofs of all of these facts may be found in [BKW].)

We now continue with the necessary definitions.

**Definition 4.2.9**

*(a) Let $G$ be a lattice ordered group and $g \in G$. Say an element $V \in \mathfrak{C}(G)$ is a value of $g$ if $V$ is maximal amongst $\mathfrak{C}(G)$ with respect to not containing $g$. For a given $g \in G$, denote the collection of all values of $g$ by $\mathcal{V}al(g)$.*

*(b) If $g \in G$ such that $|\mathcal{V}al(g)| < \infty$, then say that $g$ is finite valued.*

*(c) As a particular case of the above, if $|\mathcal{V}al(g)| = 1$ and $V$ is the value of $g$, then we say that $g$ is special, and $V$ is its special value.*

*(d) If $G$ is a lattice ordered group such that, for every $g \in G$, $g$ is finite valued, then $G$ will be called finite valued.*

*(e) If $g \in G$ such that $\{h \in G : h < g\}$ is totally ordered, then call $g$ a basic element of $G$.*

*(f) Let $g \in G$. If, for every $x \in G$, $g \wedge x = 0$ implies $x = 0$, then we say $g$ is a weak order unit.*

(g) *Say that* $g \in G$ *is a strong order unit if the convex* $l$*-subgroup generated by* $g$ *is* $G$.

(h) *Call an* $l$*-group,* $G$*, archimedean, if, whenever* $a, b \in G$*, such that* $0 < a \leq b$*, then there exists* $n \in \mathbb{N}$ *such that* $na \not\leq b$.

**Remark 4.2.10** Concerning the existence of values for a given $0 \neq g \in G$, observing that $\{0\} \in \mathfrak{C}(G)$ such that $g \notin \{0\}$, and applying Zorn's Lemma to the collection $\{C \in \mathfrak{C}(G) : g \notin C\}$, one may show that $\mathcal{V}al(g) \neq \emptyset$. In addition, employing the fact (cited in the above remarks concerning $l$-prime, convex $l$-subgroups) that $G/P$ is totally ordered if and only if $P$ is $l$-prime, one may prove that any value is $l$-prime. Then, given $g \in G$, where $g$ is special, it can be shown that the convex $l$-subgroup generated by $g$ has a unique maximal convex $l$-subgroup; in fact, this condition on $g \in G$ is equivalent to $g$ being special.([D], 27.24)

We conclude these remarks by briefly outlining the importance of values in the representation of an archimedean $l$-group possesing a weak unit, $u$. Now, given an archimedean $l$-group, $G$, with weak order unit, $u$, consider $\mathcal{V}al(u)$. Endowing $\mathcal{V}al(u)$ with the hull-kernel topology, we obtain a compact space, which in this case has the additional pleasant quality of being Hausdorff. This space is called the *Yosida space* of $G$, and is denoted by $Y(G)$. It is then possible to define an $l$-homomorphism, $\phi : G \longrightarrow D(Y(G))$, which is in fact an embedding, such that $\phi(G)$ is an $l$-subgroup of $D(Y(G))$, where $D(Y(G))$ denotes the set of all continuous functions, $f : Y(G) \longrightarrow \mathbb{R}^*$, such that $f^{-1}(\mathbb{R}) \subseteq Y(G)$ is dense open. Here, $\mathbb{R}^*$ denotes the two point compactification of $\mathbb{R}$. We also note that, in general, $D(Y(G))$ is a lattice but need not be an $l$-group itself; a full treatment of this point may be found in [BKW], and is beyond the scope of this discussion. The map $\phi$ also has the

extra qualities of carrying the weak unit, $u$, to the constant function 1, and separating the points of $Y(G)$, i.e., if $x \neq y \in Y(G)$, then there exists $g \in G$ such that $\phi(g)(x) \neq q\phi(g)(y)$. This representation is known as the *Yosida representation* of $G$.

We note, once again, proofs of all the above facts may be found in [BKW].

Realizing that the reader may now be saturated with definitions, we try to provide some firm ground to stand on by supplying a few examples of the above defined concepts.

**Example 4.2.11** Given a compact, Hausdorff space, $X$, it is elementary to see that $0 \leq f \in C(X)$ is special if and only if $|coz(f)| = 1$.

**Example 4.2.12** Again, for $X$ compact, Hausdorff, $|\mathcal{V}al(1)| < \infty$ if and only if $X$ is discrete and $|X| < \infty$.

It is well known ([BKW]) that, for an $l$-group, $G$, if $b \in G$ is basic, then it is also special. However, these two notions coincide only if the $l$-group in question is archimedean. We now provide an example which witnesses the failure of this coincidence.

**Example 4.2.13** Consider $\mathbb{R}^4$ as a group under coordinatewise addition, and order this group as follows: $(w, x; y; z) \geq (0, 0; 0; 0)$ if: (1) $z > 0$, or (2) $z = 0$ and $y > 0$, or (3) $y = 0 = z$ and $w \geq 0$ and $x \geq 0$, i.e., $\mathbb{R}^4$ is ordered coordinatewise in the first two components and by "reverse dictionary order" in the second two components. We illustrate this order graphically:

$$\begin{tikzcd} & \cdot\, 4 \arrow[d] & \\ & \cdot\, 3 \arrow[dl] \arrow[dr] & \\ 1\, \cdot & & \cdot\, 2 \end{tikzcd}$$

This ordering makes $\mathbb{R}^4$ into an $l$-group which is *not* archimedean, as, for example, $(0,0;0;0) < (1,0;0;0) \leq (0,0;1;0)$, and, for every $n \in \mathbb{N}$, we have $n(1,0;0;0) \leq (0,0;1;0)$. Now, the element $g = (0,0;0;1)$ may be seen to be a special element, as the convex $l$-subgroup it generates contains a unique maximal, convex $l$-subgroup, but is clearly not basic.

**Example 4.2.14** Consider the space $X = [0,1] \cup \{2\}$, and denote by $f$ the characteristic function $\chi_{\{[0,1]\}}$ on $[0,1]$, and by $g$ the characteristic function $\chi_{\{2\}}$ on the singleton $\{2\}$. Then $f \notin \mathfrak{m}_x = \{h \in C(X) : h(x) = 0\}$ for every $x \in [0,1]$, and so, as each $\mathfrak{m}_x \in \mathfrak{C}(C(X))$, $f$ is not special, hence, also not basic. As for $g$, it is easily seen to be special as its unique value is $\mathfrak{m}_2$, and hence $g$ is also basic as $C(X)$ is archimedean.

With the brief review of $l$-group concepts behind us we now turn to some crucial preliminary theorems regarding the structure of finite valued $l$-groups and $SV$-rings.

The following is the definitive result on the structure of finite valued $l$-groups. It was proved by Conrad and may be found in [D], 46.10.

**Theorem 4.2.15** *(Conrad's Finite Basis Theorem): For an $l$-group, $G$, the following are equivalent:*

*(a) $G$ has a finite maximal collection of pairwise disjoint basic elements, i.e., $G$ has a finite basis.*

*(b) Every disjoint subset of $G$ is finite.*

*(c) $|Min_l(G)| < \infty$.*

*(d) $Min_l(G) \subseteq \mathcal{P}_r(G)$.*

In addition, any of the above conditions imply $G$ is finite valued, where, if $n$ is the size of a finite basis for $G$, then there does not exist any collection of $n + 1$ pairwise disjoint elements of $G$.

The next result is also due to Conrad, and characterizes those elements of an $l$-group, $G$, which are basic.

**Theorem 4.2.16** *Let $G$ be an $l$-group and $0 \leq g \in G$. Then the following are equivalent:*

*(a) $g$ is basic;*

*(b) the convex $l$-subgroup generated by $g$ is totally ordered;*

*(c) $g^{\perp\perp}$ is a minimal polar;*

*(d) $g^{\perp\perp}$ is the largest totally ordered, convex $l$-subgroup of $G$ containing $g$.*

Having amassed a sufficient amount of information regarding finite valued $l$-groups, it is now possible to consider the connection between the concept of an $f$-ring of finite rank and that of a finite valued $f$-ring. We therefore supply the central definitions of this notion.

**Definition 4.2.17**

(a) *Given an $f$-ring, $A$, and $\mathfrak{m} \in Max(A)$, the rank of $A$ at $\mathfrak{m}$, denoted by $rk(A, \mathfrak{m})$, is the number of minimal prime ideals of $A$ contained in $\mathfrak{m}$, if finite, or is $\infty$ otherwise.*

(b) *Let $A$ be an $f$-ring. The rank of $A$, denoted by $rk(A)$, is given by $rk(A) =$* $\sup_{\mathfrak{m} \in Max(A)}(\mathrm{rk}(A, \mathfrak{m}))$.

(c) *For a Tychonoff space, $X$, the rank of $X$, denoted by $rk(X)$, is given by*

$$\mathrm{rk}(X) = \mathrm{rk}(C(X)).$$

In an attempt to animate the above definition for the reader, three theorems will be provided which place it in the setting of $SV$-rings. First, we must remind the reader of the concept of a uniformly complete $f$-ring.

**Definition 4.2.18**

(a) *Let $A$ be an $f$-ring and $\{a_n\}_{n<\omega}$ a sequence in $A$. Then $\{a_n\}_{n<\omega}$ is said to be uniformly Cauchy if, for every $k \in \mathbb{N}$, there exists $N(k) \in \mathbb{N}$, such that, for all $m, l \geq N(k)$, we have $k|a_l - a_m| \leq 1$.*

(b) *Again, let $A$ be an $f$-ring and $\{a_n\}_{n<\omega}$ a sequence in $A$. The sequence $\{a_n\}_{n<\omega}$ is said to converge uniformly to $a \in A$ if, for every $k \in \mathbb{N}$, there exists an $N(k) \in \mathbb{N}$, such that, for every $m \geq N(k)$, we have $k|a_m - a| \leq 1$.*

(c) *An $f$-ring, $A$, is called uniformly complete if every uniformly Cauchy sequence in $A$ converges uniformly to a unique $a \in A$.*

**Remark 4.2.19** It is shown in [HdP] that a uniformly complete $f$-algebra with identity (here, by $f$-algebra we mean an $f$-ring which is also a real vector lattice) is archimedean, has bounded inversion, and is square root closed.

And now the promised triple of theorems ([HLMW]):

**Theorem 4.2.20** *Every uniformly complete $SV$-algebra with identity has finite rank.*

**Theorem 4.2.21** *If $A$ is a uniformly complete $f$-algebra with identity, such that for every $\mathfrak{m} \in Max(A)$, $rk(A, M) < \infty$, then $rk(A) < \infty$.*

**Theorem 4.2.22** *Let $X$ be a compact, Hausdorff space. Then:*

*(a) For $X$ an $SV$-space, we have $rk(X) < \infty$*

*(b) If for every $x \in X$ we have $rk(C(X), \mathfrak{m}_x) < \infty$, then $rk(X) < \infty$*

*(c) If $X$ is an $SV$-space, then there exists $k \in \mathbb{N}$ such that, for every $n > k$, any collection of $n$ pairwise disjoint cozero sets have disjoint closures.*

**Remark 4.2.23** We wish to point out that it is in 4.2.22 (c) above that the promised connection between $SV$-spaces and rings of finite rank, and finite valued $f$-rings is realized. As for the proofs of the above theorems, all may be found in [HLMW].

To assist the reader in developing some feel for compact $SV$-spaces, the following two contrasting examples are provided:

**Example 4.2.24** As noted in 4.2.5, any Bézout $f$- ring is an $SV$-ring. Hence, by 4.1.4, it is immeadiate that any compact $F$ -space is a compact $SV$-space.

**Example 4.2.25** Consider $\alpha\mathbb{N}$, the one point compactification of $\mathbb{N}$ endowed with the discrete topology. Partition $\mathbb{N}$ into a countable number of pairwise disjoint, infinite subsets, say $\{S_n\}_{n<\omega}$ and define $f_n(x_m) = 1/m$, for $x_m \in S_n$, and $f_n[(\alpha\mathbb{N}) - S_n] = 0$. Then $S_n = coz(f_n)$, hence $\{S_n\}_{n<\omega}$ is an infinte collection of pairwise disjoint, cozero sets of $\alpha\mathbb{N}$ such that $\cap_{n<\omega}(cl_{\alpha\mathbb{N}}(S_n)) = \{\alpha\}$. It follows from (c) of the previous theorem that $\alpha\mathbb{N}$ is not an $SV$-space.

All that is needed at this point to position ourselves to prove the main theorem are two lemmas concerning the flat dimension of certain $l$-ideals of semiprime, and local semiprime $f$-rings. First, an important technical result is stated regarding local $SV$-rings. Its proof may be found in [HLMW].

**Theorem 4.2.26** *Let $A$ be a local semiprime $f$-ring of finite rank with identity and bounded inversion which is square root closed. Then $A$ is an $SV$-ring if and only if, whenever $0 \leq a \leq b \in A$, and $b$ is special, then there is an $x \in A$ such that $a = bx$.*

Now for the two lemmas.

**Lemma 4.2.27** *Let $A$ be a 2-convex, square root closed, semiprime $f$-ring. Then every polar of $A$ is flat.*

**Proof:** This is immediate from 3.1.6, as every polar of $A$ is semiprime. In fact, let $P \in \mathcal{P}(A)$, $b \in P^{\perp}$ and $0 \leq a \in A$ such that $a^2 \wedge b = 0$. Then, as $A$ is an $f$-ring we have $0 = a^2 \wedge b = (aa) \wedge b = a(a \wedge b)$. Since $A$ is semiprime and $0 = a(a \wedge b)$, it follows that $0 = a \wedge (a \wedge b) = ((a \wedge a) \wedge b) = a \wedge b$, i.e., $a \in P^{\perp\perp} = P$. Hence, $P$ is semiprime. $\square$

**Lemma 4.2.28** *Let $A$ be a local semiprime $SV$-ring of finite rank with identity and bounded inversion which is square root closed. If $I$ is a totally ordered $l$-ideal of $A$ which is finitely generated as an ideal, then $fd(I) \leq 1$.*

**Proof:** Let $I = (a_1, ..., a_n)$, the ideal generated by $a_i$, $1 \leq i \leq n$. We may assume each $a_i$ is positive, since $I$ is totally ordered, and, that there exists an $i_0$, $1 \leq i_0 \leq n$, such that $a_{i_0} = \vee_{i=1}^{n} a_i$; define $a = a_{i_0}$. As $\mathrm{rk}(A) < \infty$, $A$ is local, and $Min(A) = Min_l(A)$, it follows that $|Min_l(A)| < \infty$. Hence, by Conrad's Finite Basis Theorem, $A$ is a finite valued $l$-group. By [C], or else [D], $a = \sum_{j=1}^{m} s_j$, where $\{s_j\}_{j=1}^{m}$ is a

collection of pairwise disjoint, special elements of $A$. Then, since $I$ is an $l$-ideal, and by the pairwise disjointness of the $s_j$, we have $0 \leq |s_{j_0}| \leq \sum_{j=1}^m |s_j| = |\sum_{j=1}^m s_j| = |a|$; thus $s_j \in I$ for every $j$, $1 \leq j \leq m$. However, recalling again that $I$ is totally ordered, and by the fact that the $s_j$ are pairwise disjoint, we conclude that only one $s_j$ is nonzero. Therefore, $a$ itself may be taken to be special. We claim that $I$ is then generated as an ideal by the single element $a$. In fact, $a$ was chosen so that $a_i \leq a$ for each $i$, $A$ satisfies the requirements of 4.2.9(c), and $a$ is special; we may therefore conclude that, for every $i$, there exists a $c_i \in A$ with $a_i = ac_i$; consequently, $I$ is the principal ideal generated by $a$. Hence there exists a short exact sequence of $A$-modules:

$$0 \longrightarrow a^{\perp} \longrightarrow A \longrightarrow I \longrightarrow 0$$

As $A$ is obviously flat (in fact free) over itself as an $A$-module, invoking 1.4(iii) and 4.2.27, we conclude $fd(I) \leq 1$. □

We now arrive at the main result of this chapter.

**Theorem 4.2.29** *Let $A$ be a 2-convex, square root closed, semiprime $SV$-ring with bounded inversion. Then $wD(A) \leq \mathrm{rk}(A)$.*

**Proof:** The proof proceeds by induction on $\mathrm{rk}(A)$. For the case $\mathrm{rk}(A) = 1$, every maximal ideal of $A$ contains a unique minimal prime ideal $P$. By [MW], $P$ has the form $\mathcal{O}(\mathfrak{m})$ where $\mathfrak{m}$ is the unique maximal ideal containing $P$, and $A/\mathcal{O}(\mathfrak{m}) \cong A_{\mathfrak{m}}$ is a valuation domain. Hence, by 3.1.14, $wD(A) \leq 1 = \mathrm{rk}(A)$.

Now, assume the result is true for all $k < n$, and let $A$ satisfy the conditions of the theorem where $\mathrm{rk}(A) = n$. Let $\mathfrak{m} \in Max(A)$ and consider the localization $A_{\mathfrak{m}}$. As $\mathrm{rk}(A) \leq n$, $A_{\mathfrak{m}}$ is a finite valued $l$-group with a basis $\mathcal{B} = \{b_1, ...b_l\}$, where $l \leq n$. Pick $0 \leq b \in \mathcal{B}$ and consider the principal polar $P = b^{\perp\perp}$. By 4.2.16(d), $P$ is a totally

ordered, convex $l$-subgroup of $A_{\mathfrak{m}}$, and is even an $l$-ideal by 4.2.10. Now, let $I$ be an arbitrary ideal of $A_{\mathfrak{m}}$. We must prove that $fd(I) \leq n-1$; for then, by 1.4(iii), it will follow that $wD(A_{\mathfrak{m}}) \leq n = \mathrm{rk}(A)$. To this end, note that $I \cap P$ is a convex $l$-subgroup of $A$. In fact, if $c \in I \cap P$ and $d \in A_{\mathfrak{m}}$ such that $0 \leq |d| \leq |c|$, it is the case that $d^+, d^- \leq |c|$. As $|c| \in P$, a totally ordered group, we have $|c|$ is special in $A_{\mathfrak{m}}$, and so there exist $x_0, x_1 \in A_{\mathfrak{m}}$ such that $d^+ = |c|x_0$ and $d^- = |c|x_1$; consequently, $|d| = d^+ + d^- = |c|(x_0 + x_1) \in I \cap P$, i.e., $I \cap P$ is a convex $l$-subgroup, as claimed. Next, consider the factor ring $A_{\mathfrak{m}}/P$ along with the canonical ring homomorphism $\phi : A_{\mathfrak{m}} \longrightarrow A_{\mathfrak{m}}/P$. As proved in 3.2.14,

$$fd_{A_{\mathfrak{m}}}((I+P)/P) \leq fd_{A_{\mathfrak{m}}/P}((I+P)/P) + fd_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}/P).$$

Note that

$$0 \longrightarrow P \longrightarrow A_{\mathfrak{m}} \longrightarrow A_{\mathfrak{m}}/P \longrightarrow 0$$

is an exact sequence of $A_{\mathfrak{m}}$-modules, hence $fd_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}/P) \leq 1$, as $A_{\mathfrak{m}}$ and $P$ are flat $A_{\mathfrak{m}}$-modules. Note also, that $A_{\mathfrak{m}}/P$ is an $SV$-ring satisfying the conditions of the theorem with $\mathrm{rk}(A_{\mathfrak{m}}/P) \leq n-1 < n$. Thus, by the induction hypothesis,

$$fd_{A_{\mathfrak{m}}}((I+P)/P) \leq n-1.$$

Since $(I+P)/P \cong I/(I \cap P)$, as $A_{\mathfrak{m}}$-modules, we may conclude that

$$fd_{A_{\mathfrak{m}}}(I/(I \cap P)) \leq n-1.$$

Now,

$$0 \longrightarrow (I \cap P) \longrightarrow I \longrightarrow I/(I \cap P) \longrightarrow 0$$

is an exact sequence of $A_{\mathfrak{m}}$-modules. Hence, for every $A_{\mathfrak{m}}$-module, $B$, there exists a long exact sequence in homology

$$\cdots \longrightarrow Tor_{n+1}(I/(I \cap P), B) \longrightarrow Tor_n(I \cap P, B) \longrightarrow$$

$$Tor_n(I, B) \longrightarrow Tor_n(I/(I \cap P), B) \longrightarrow \cdots$$

By 4.2.28, $fd_{A_\mathfrak{m}}(I \cap P) \leq 1$, and, as mentioned above,

$$fd_{A_\mathfrak{m}}(I/(I \cap P)) \leq n - 1.$$

Therefore, by 3.2.7, $fd_{A_\mathfrak{m}}(I) \leq n - 1$, and thus we have shown $wD(A_\mathfrak{m}) \leq n$ as promised. Invoking 1.4(iii), we obtain $wD(A) \leq n = \mathrm{rk}(A)$. $\square$

# CHAPTER 5
## Cardinal Functions on Stone Spaces and $P$-Spaces

This chapter signals the begining of our investigation into the global dimension of the ring of continuous functions, $C(X)$. It is here that we define the cardinal functions which will form the basic topological tools for this endeavor. The first section will provide a brief review of the concept of Stone duality between boolean algebras and compact, zero dimensional, Hausdorff spaces. For details regarding this topic, the reader is referred to [Ko], Chapter 4. This duality will then be applied to a particular class of spaces in order to furnish the necessary bridge connecting the homological concepts outlined in Chapter 3, and the topological results which are the focus of the second section of this chapter.

### 5.1 P-Spaces

We begin with a definition.

**Definition 5.1.1** *Let $X$ be a space. Then $X$ is said to be zero dimensional if there exists a base for the topology on $X$ which consists of clopen subsets of $X$.*

If one is given a zero dimensional space, $X$, then under the usual set theoretic operations of union, intersection, and complementation, the collection of all clopen subsets of $X$, denoted by $\mathfrak{B}(X)$, forms a boolean algebra. In addition, if $X$ and $Y$ are zero dimensional spaces, and $f : X \longrightarrow Y$ is a continuous function, then we may define a boolean homomorphism, $\phi_f : \mathfrak{B}(Y) \longrightarrow \mathfrak{B}(X)$ by $\phi_f(B) = f^{-1}(B)$. It is elementary, by the properties of continuous functions, that the inverse image of a

clopen subset of $Y$ is a clopen subset of $X$; as for the preservation of the boolean operations between $\mathfrak{B}(Y)$ and $\mathfrak{B}(X)$, this is immeadiate from the properties of inverse images of functions. Specializing to the category $\mathcal{KZH}$, whose objects consist of all compact, zero dimensional, Hausdorff spaces, and whose morphisms are all continuous maps between such spaces, one may define a contravariant functor, $\mathcal{B}$, from $\mathcal{KZH}$ to the category $\mathcal{B}ool$, whose objects consist of all boolean algebras, and whose morphisms are all boolean homomorphisms. This functor operates on spaces, $X$, by $\mathcal{B}(X) = \mathfrak{B}(X)$, and on continuous functions, $f$, by $\mathcal{B}(f) = \phi_f$, as above. Conversely, given a boolean algebra, $B$, one would like to associate an object in $\mathcal{KZH}$ with $B$. This may be achieved by recalling that, in particluar, a boolean algebra is a lattice under the given boolean operations $\vee$ and $\wedge$. Then, endowing the collection of all ultrafilters on $B$, $Ult(B)$, with the hull-kernel topology as outlined in 1.2, one indeed obtains a compact, zero dimensional, Hausdorff space, which we denote by $\mathcal{S}(B)$; this space is customarily called the *Stone dual* of the boolean algebra $B$. To complete the definition of the desired functor from $\mathcal{B}ool$ to $\mathcal{KZH}$, consider a boolean homorphism, $\phi$, from a boolean algebra $B$ to a boolean algebra $C$. As the functor we are defining will also be contravariant, since $\mathcal{B}$ is, the domain of $\mathcal{S}(\phi)$ must be $\mathcal{S}(C)$. Therefore, let $x \in \mathcal{S}(C)$, and recall that $x$ is actually an element of $Ult(C)$. Then $\phi^{-1}[x] = \{\phi^{-1}(c) : c \in x\} \in Ult(B)$; hence, defining $\mathcal{S}(\phi)(x) = \phi^{-1}[x]$, we have a function from $Ult(C)$ to $Ult(B)$. It is routine to verify that this function is indeed a morphism in $\mathcal{KZH}$, i.e., that it is continuous. It should be noted that if $B$ is a boolean algebra, then $\mathcal{B}(\mathcal{S}(B))$ is a boolean algebra which is naturally boolean isomorphic to $B$; conversely, if $X$ is a compact, zero dimensional, Hausdorff space, then $\mathcal{S}(\mathcal{B}(X))$ is naturally homeomorphic to $X$, i.e., given $\phi : B \longrightarrow C$ and $f : X \longrightarrow Y$, where $B, C$ are boolean algebras, $\phi$ is a boolean homomorphism, and $X, Y$ are compact, zero dimensional,

Hausdorff spaces, with $f$ a continuous map between them, then the following diagrams commute,

$$\begin{tikzcd} B \arrow[r, "\iota_B"] \arrow[d, "\phi"'] & \mathcal{B}(\mathcal{S}(B)) \arrow[d, "\mathcal{B}(\mathcal{S}(\phi))"] \\ C \arrow[r, "\iota_C"] & \mathcal{B}(\mathcal{S}(C)) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\iota_X"] \arrow[d, "f"'] & \mathcal{S}(\mathcal{B}(X)) \arrow[d, "\mathcal{S}(\mathcal{B}(f))"] \\ Y \arrow[r, "\iota_Y"] & \mathcal{S}(\mathcal{B}(Y)) \end{tikzcd}$$

where $\iota_B$, $\iota_C$, $\iota_X$, and $\iota_Y$, are the isomorphisms mentioned above.

At this point we shall define the principal class of spaces which will occupy our investigation for the duration of this dissertation; namely the class of $P$-spaces. However, at times, when no further effort is required, we will state appropriate results for the larger class of zero dimensional spaces. First, we briefly motivate our restriction to this class.

Recall that in 3.1.4 a result of Osofsky was cited which provides an upper bound for the global dimension of a ring in terms of the cardinality of generating sets for ideals of the ring. Specifically, if $A$ is a ring in which every ideal of $A$ is generated by at most $\aleph_n$ many elements, then

$$gD(A) \leq wD(A) + n + 1.$$

Thus, it seems reasonable to begin an investigation into the global dimension of rings of continuous functions by first restricting to a class of spaces, $\mathfrak{X}$, so that $wD(C(X)) = 0$ if $X \in \mathfrak{X}$. Now, by 1.4(iv), the condition $wD(A) = 0$ for a ring, $A$, is equivalent to the requirement that $A$ be von Neumann regular; hence, as it is well known ([GJ], Chapter 14) that $C(X)$ is von Neumann regular if and only if $X$ is a $P$-space, we are well justified in our choice of restriction.

Although any of the following in the list of equivalent conditions may be taken as a definition of a $P$-space, the author believes that the most natural definition is given by requiring that the topology for the open sets on the space be closed under countable intersections.

**Definition 5.1.2** *Call a space, $X$, a $P$-space if any of the following equivalent conditions hold:*

*(a) every countable intersection of open sets is open;*

*(b) every zero set is open;*

*(c) $C(X)$ is von Neumann regular;*

*(d) every ideal of $C(X)$ is generated by idempotents;*

*(e) for every $\mathfrak{m} \in Max(C(X))$, $\mathcal{O}(\mathfrak{m}) = \mathfrak{m}$.*

The equivalence of the above conditions may be found in [GJ], Chapter 14.

As a note to the above definition, recall that a base for the topology on a Tychonoff space is given by the collection of cozero sets of functions $f \in C(X)$. Thus, if $X$ is a $P$-space, since every zero set is open (and is indeed closed as it is the inverse image of a point in $\mathbb{R}$ under a continuous function) it is a clopen set in $X$. Therefore, since cozero sets are complements of zero sets, if $X$ is a $P$-space, it follows that the cozero sets on $X$ form a clopen base for the topology on $X$, and so $X$ is zero dimensional. In fact, if $X$ is a $P$-space, $X$ is more than just zero dimensional. Indeed, since every $Z \in \mathcal{Z}[X]$ is clopen, and, for every clopen set $B \in \mathfrak{B}(X)$, we have $B = Z(e_{(X-B)})$, (where $e_{(X-B)} \in C(X)$ denotes the characteristic function of $X - B$) it follows that $\mathcal{Z}[X] = \mathfrak{B}(X)$. From this observation it is elementary to prove that a $P$-space is *strongly zero dimensional*, i.e.,

**Definition 5.1.3** *A space, $X$, is said to be strongly zero dimensional if, for every pair $Z_1, Z_2 \in \mathcal{Z}[X]$, there exists a $B \in \mathfrak{B}(X)$ such that,*

$$Z_1 \subseteq B \subseteq X - Z_2.$$

We pause here to introduce a canonical method of constructing a $P$-space from a given space $X$. As $P$-spaces have not to this point, (excluding [GH2]), been a central object of study in the theory of $C(X)$, this construction is offered in an attempt to enlarge the collection of well known examples of $P$-spaces. To this end, given a space, $X$, define a new space, $X_\delta$ , so that the underlying sets of $X$ and $X_\delta$ agree. As for a base for the open sets on $X_\delta$, this is obtained by taking all countable intersections of basic open sets on $X$. This new space $X_\delta$ is a $P$-space. The sense in which this construction is canonical is that what we have actually done is define a covariant functor, $\delta(\cdot)$, from the category of all Tychonoff spaces to the category of all $P$-spaces. In fact, this functor is what is known in the theory as a *coreflection.* In other words, if $X$ is a space, then there exists a map $\delta : X_\delta \longrightarrow X$, so that if $Y$ is any other $P$-space, and $f : Y \longrightarrow X$ is any other map, then there exists a unique map $\hat{f} : Y \longrightarrow X_\delta$, such that $\delta\hat{f} = f$. For the sake of completeness, we provide two examples below.

**Example 5.1.4** Applying the functor, $\delta(\cdot)$ to the space of real numbers, $\mathbb{R}$, we obtain the object $(\mathbb{R})_\delta$, a discrete $P$-space of cardinality $\mathfrak{c}$.

**Example 5.1.5** Consider $\alpha(D(\aleph_n))$, the one point compactification of the discrete space of cardinality $\aleph_n$. Applying the $P$-space coreflection to $\alpha(D(\aleph_n))$, the resulting object, $(\alpha(D(\aleph_n)))_\delta$, is a Lindelöf space. Specifically, the open sets for the topology on $(\alpha(D(\aleph_n)))_\delta$ consist of all points of $D$, and all subsets of $\alpha(D(\aleph_n))$ whose complements in $D$ are countable.

And now a final comment pertaining to the equality $\mathcal{Z}[X] = \mathfrak{B}(X)$, for $X$ a $P$-space. As noted in 1.2, the Stone-Čech compactification, $\beta X$, of a Tychonoff space, $X$, may be described as the space of ultrafilters on the lattice $\mathcal{Z}[X]$ equipped with the hull-kernel topology. Due to the fact that the set theoretic operations of union, intersection, and complementation make $\mathcal{Z}[X]$ into a boolean algebra when $X$ is a $P$-space, the equality cited above is not only one of sets, but also one of boolean algebras. Then, recalling that the Stone dual of a boolean algebra, $B$, is the collection of ultrafilters, $Ult(B)$, topologized by the hull-kernel topology, then, for a given $P$-space, $X$, we obtain the homeomorphism $\beta X \cong \mathcal{S}(\mathcal{Z}[X]) \cong \mathcal{S}(\mathfrak{B}(X))$; i.e., the Stone-Čech compactification of a $P$-space, $X$, may be viewed as the Stone dual of the boolean algebra of clopen subsets of $X$. In light of this discussion, and the central role played by $\beta X$ in the theory of $C(X)$, we state here a full characterization of $\beta X$.

**Theorem 5.1.6** ([GJ], 6.5)*Let $X$ be a Tychonoff space. Then $\beta X$, the Stone-Čech compactification of $X$, is the unique compactification of $X$, up to homeomorphism, with the following equivalent properties:*

*(a) $X$ is $C^*$-embedded in $\beta X$;*

*(b) distinct ultrafilters on $\mathcal{Z}[X]$ have distinct limits in $\beta X$;*

*(c) for every pair $Z_1, Z_2 \in \mathcal{Z}[X]$, we have*

$$cl_{\beta X}(Z_1) \cap cl_{\beta X}(Z_2) = cl_{\beta X}(Z_1 \cap Z_2);$$

*(d) for every disjoint pair of zero sets, $Z_1, Z_2 \in \mathcal{Z}[X]$, we have*

$$cl_{\beta X}(Z_1) \cap cl_{\beta X}(Z_2) = \emptyset;$$

(e) *for every compact space $K$, and continuous map $\psi : X \longrightarrow K$, there exists a unique extension $\Psi : \beta X \longrightarrow K$, such that $\Psi\iota = \psi$, where $\iota : X \longrightarrow \beta X$ is the canonical embedding of $X$ into $\beta X$.*

## 5.2 Cardinal Functions

With the machinery in place to translate between topological conditions on $\beta X$, and the boolean algebra, $\mathfrak{B}(X)$, of clopen subsets of $X$, when $X$ is a $P$-space, it is now time to define the necessary cardinal functions which will eventually provide upper and lower boundes for the global dimension of the ring $C(X)$. First, we wish to remind the reader that all of our infinite cardinals will be of the form $\aleph_n$, for $n \in \mathbb{N}$, unless otherwise specified. Recall that this assumption was made, in part, to avoid *sup vs. max* problems, i.e., to avoid the case where a cardinal function takes on an infinte value, but this value is not actually realized by a collection of objects of the space in question. Although suchphenomena are interesting in their own right, they are an unnecessary distraction to this dissertation. Hence, we will assume that all cardinal functions are defined on spaces only when the property in question is realized within that space. In addition, all of the cardinal functions defined will take on values greater than or equal to $\aleph_0$, hence, all cardinal functions, $\phi$, will be assumed to have the form $\phi + \aleph_0$, although the addition of the $\aleph_0$ will not appear in the definitions. As a general reference for cardianl functions, the reader is directed to [En].

Throughout, $X$ is a Tychonoff space, as usual.

**Definition 5.2.1** *The density character of $X$, denoted by $d(X)$, is given by*

$$d(X) = \inf\{\,\kappa : \text{there exists}\, Y \subseteq X \,\text{such that}\, Y \,\text{is dense in}\, X \,\text{and}\, |Y| = \kappa\,\}.$$

**Remark 5.2.2**

*(a)* If $X$ is in $\mathcal{KZH}$, then $|\mathfrak{B}(X)| = 2^{d(X)}$. In fact, if $B \in \mathfrak{B}(X)$, then the map $\phi : \mathfrak{B}(X) \longrightarrow \mathcal{P}(Y)$ defined by $\phi(B) = B \cap Y$ is surjective, since $X$ is compact, and is injective, by the zero dimensionality of $X$.

*(b)* When $X$ is a space such that $d(X) = \aleph_0$, then $X$ is commonly called *separable.*

*(c)* Letting $D$ denote the two point, discrete space, then, for an infinte cardinal, $\kappa$, $D^\kappa$ denotes the generalized Cantor space of cardinality $2^\kappa$. This space may be viewed as the product of $D$ over an indexing set of size $\kappa$, endowed with the Tychonoff, i.e., the product, topology. For any $D^\kappa$, it can be shown, ([Ef]), that $d(D^\kappa)$ =log($\kappa$), where, for any infinite cardinal, $\kappa$, log($\kappa$) =inf$\{\lambda : 2^\lambda \geq \kappa\}$. For example, $d(D^{\aleph_0}) = \aleph_0 = d(D^\mathfrak{c})$, where, $\mathfrak{c} = 2^{\aleph_0} = |\mathbb{R}|$. Note also, that if CH, the continuum hypothesis, is denied, then for every $\aleph_0 < \kappa < 2^{\aleph_0}$, we have $\log(\kappa) = \aleph_0$.

**Definition 5.2.3** *The weight of a space, X, denoted by w(X), is defined by*

$$w(X) = \inf\{\kappa : \text{there exists}, \mathfrak{B}, \text{a base for the open sets on } X, \text{ with } |\mathfrak{B}| = \kappa\}.$$

**Remark 5.2.4**

*(a)* For any space, $X$, the inequality $d(X) \leq w(X)$ always holds.

*(b)* For a space, $X$, in $\mathcal{KZH}$, we always have $w(X) = |\mathfrak{B}(X)| = 2^{d(X)}$.

*(c)* Given the space $D^\mathfrak{c}$, and $\mathfrak{c} = \aleph_n$, then

$$w(D^\mathfrak{c}) = \aleph_n > \aleph_0 = \log(\mathfrak{c}) = d(D^\mathfrak{c});$$

hence, the spread between $w(D^\mathfrak{c})$ and $d(D^\mathfrak{c})$ may be as large as one wishes, depending on the set theory one assumes, i.e., for what choice of $n \in \mathbb{N}$ we have $\mathfrak{c} = \aleph_n$.

*(d)* If $X$ is a $P$-space, then $\beta X = \mathcal{S}(\mathfrak{B}(X))$, hence

$$w(\beta X) = w(\mathcal{S}(\mathfrak{B}(X))) = |\mathfrak{B}(X)| = 2^{d(X)}.$$

Therefore, since $\beta X$ is a compactification of $X$,

$$d(\beta X) \leq d(X) \leq w(X) \leq |\mathcal{Z}[X]| = |\mathfrak{B}(X)| = 2^{d(X)},$$

and so $d(\beta X) < w(\beta X)$.

**Definition 5.2.5** *Given a space, $X$, we say $\mathfrak{U}$ is a cellular family in $X$, if $\mathfrak{U}$ is a family of pairwise disjoint, open subsets of $X$.*

**Definition 5.2.6** *Let $X$ be a space and define the cellularity of $X$, denoted $c(X)$, by*

$$c(X) = \sup\{\,\kappa\, :\, \text{there exists}\,, \mathfrak{U},\, \text{a cellular family in}\, X,\, \text{with}\, |\mathfrak{U}| = \kappa\,\}.$$

**Remark 5.2.7**

*(a)* For any space, $X$, the inequality $c(X) \leq d(X) \leq w(X)$ holds. If, in addition, $X$ is metrizable, then $c(X) = d(X) = w(X)$.

*(b)* If $X$ is a space such that $c(X) = \aleph_0$, then $X$ is commonly said to satisfy the *countable chain condition*, or in other words, $X$ is a *c.c.c* space.

*(c)* The spread between $c(X)$ and $d(X)$ may also be as large as one wishes. Note, for the space $D^\kappa$, where $\kappa = 2^\lambda$ and $\lambda > \aleph_0$, then

$$c(D^\kappa) = \aleph_0 < d(D^\kappa) = \log(\kappa) = \lambda.$$

**Definition 5.2.8** *Let $X$ be a space and $\mathfrak{U}$ be a collection of pairwise disjoint, open subsets of $X$. Say that $\mathfrak{U}$ is a separated cell in $X$, if, for any $\mathfrak{T} \subseteq \mathfrak{U}$, there exists $f \in C(X)$, dependent on $\mathfrak{T}$, such that $f[U] = 1$ for every $U \in \mathfrak{T}$, and $f[U] = 0$ for every $U \in (\mathfrak{U} - \mathfrak{T})$.*

**Definition 5.2.9** *For a space, $X$, the separated cellularity of $X$, denoted $sc(X)$, is defined by*

$$sc(X) = \sup\{\,\kappa\,:\,\text{there exists}\,,\mathfrak{U},\ \text{a separated cell in}\ X,\ \text{with}\ |\mathfrak{U}| = \kappa\,\}.$$

**Remark 5.2.10**

*(a)* For any space, $X$, the inequalities

$$sc(X) \leq c(X) \leq d(X) \leq w(X)$$

hold. Considering the space $(\alpha(D(\aleph_n)))_\delta$ from 5.1.5. For ease of notation, from this point on this space will be denoted by $\lambda(D(\aleph_n))$. We have $sc(\lambda(D(\aleph_n))) = \aleph_0$, while $c(\lambda(D(\aleph_n))) = \aleph_n$, hence, again the spread between the cellularity of a space, $X$, and its separated cellularity may be as large as one likes.

*(b)* For $X$, a strongly zero dimensional space, if there exists, $\mathfrak{U}_0$, a separated cell in $X$, then, clearly, as $X$ has a base for the open sets consisting of clopen sets, there exists a separated cell, $\mathfrak{U}_1$, in $X$ whose members are clopen sets. Now, if $\mathfrak{U}$ is a separated cell in $X$ consisting of clopen sets, then $\mathfrak{U}_\beta = \{\,cl_{\beta X}(U)\,:\,U \in \mathfrak{U}\}$ is a cellular family of clopen sets in $\beta X$. In addition, as $X$ is $C^*$-embedded in $\beta X$, if $\mathfrak{T}_\beta \subseteq \mathfrak{U}_\beta$, then there exists $f \in C^*(X)$ satisfying the condition that $f[U \cap X] = 1$ for every $U \in \mathfrak{T}_\beta$, and $f[U \cap X] = 0$ for every $U \in (\mathfrak{U}_\beta - \mathfrak{T}_\beta)$; consequently, there exists $\hat{f} \in C(\beta X)$ such that $\hat{f}[U] = 1$ for every $U \in \mathfrak{U}_\beta$, and $\hat{f}[U] = 0$ for every $U \in (\mathfrak{U}_\beta - \mathfrak{T}_\beta)$, i.e., $\mathfrak{U}_\beta$ is a separated cell in $\beta X$. The converse of this argument is trivial as a cellular family in $\beta X$, when restricted to $X$ is again a cellular family, and the restriction of a continuous function on $\beta X$ to $X$, is again continuous on $X$. The upshot of all of this is that $sc(X) = sc(\beta X)$, when $X$ is strongly zero dimensional.

In an effort to further illuminate the concept of the separated cellularity of a compact space, $X$, a theorem which characterizes when a compact space has a separated cell of size $\kappa$, is supplied. The reader is reminded that a closed subset, $C$, of a space $X$, is said to be *regular closed* in $X$, if $cl_X(int_X(C)) = C$. The following theorem appears in [FMMc2].

**Theorem 5.2.11** *Let $X$ be a compact space. Then the following are equivalent:*

*(a) sc(X)=$\kappa$;*

*(b) for every infinite cardinal, $\lambda \leq \kappa$, there exists a regular closed subset, $C$, of $X$, and a continuous function, $f : C \longrightarrow \beta D(\lambda)$, such that $f$ is a surjection, where $\beta D(\lambda)$ is the Stone-Čech compactification of the discrete space of cardinality $\lambda$;*

*(c) for every infinite cardinal, $\lambda \leq \kappa$, there exists a regular closed subset, $C$, of $X$, and a continuous function, $f : C \longrightarrow \beta D(\lambda)$, such that $f$ is a retraction onto $\beta D(\lambda)$, i.e., there exists $\iota : \beta D(\lambda) \longrightarrow C$, a continuous injection, so that $f\iota = id_{\beta D(\lambda)}$.*

For the sake of completeness, we recall the following definition from Chapter 3.

**Definition 5.2.12** *Let $(\mathfrak{B}, \vee, \wedge, (\cdot)')$ be a boolean algebra. Call a subset, $\mathfrak{F}$, of $\mathfrak{B}$, independent, if for every pair $F, G \in [\mathfrak{F}]^{<\omega}$, with $F \cap G = \emptyset$, it is the case that $(\wedge F) \wedge (\wedge G') \neq 0$, where $[\mathfrak{F}]^{<\omega}$ denotes the collection of all finite subsets of $\mathfrak{F}$, and $G'$ is the collection of all boolean complements of elements of $G$.*

**Remark 5.2.13** Note that $\mathfrak{F} \subseteq \mathfrak{B}$ is independent if and only if the boolean subalgebra of $\mathfrak{B}$ generated by $\mathfrak{F}$ is free. Conversely, any free boolean subalgebra of $\mathfrak{B}$ of cardinality $\kappa$ is generated by an independent subset $\mathfrak{F}$ of $\mathfrak{B}$, where $|\mathfrak{F}| = \kappa$.

**Definition 5.2.14** *Let $\mathfrak{B}$ be a boolean algebra. Define the independence character of $\mathfrak{B}$, denoted $ind(\mathfrak{B})$, by*

$$ind(\mathfrak{B}) = \sup\{\, \kappa : \text{there exists}\, \mathfrak{F}, \text{ independent in}\, \mathfrak{B}, \text{ where}\, |\mathfrak{F}| = \kappa \,\}.$$

By an abuse of notation we arrive at the following definition.

**Definition 5.2.15** *Let $X$ be a zero dimensional space, and define the independence character of $X$, by $ind(X)$=$ind(\mathfrak{B}(X))$.*

**Remark 5.2.16** For a $P$-space, $X$, as $\mathfrak{B}(X) \cong \mathfrak{B}(\beta X)$ under the boolean homomorphism $\phi(B) = cl_{\beta X}(B)$, it is immediate that $ind(X) = ind(\beta X)$.

At this point we remark on a number of techniques that will be discussed regarding the construction of independent families in $\mathfrak{B}(X)$, for $X$ a zero dimensional space. First, a result by Hausdorff concerning the existence of independent families in a power set algebra is needed.

**Theorem 5.2.17** *Let $Y$ be a set such that $|Y| = \kappa$. Then there exists $X \subseteq \mathcal{P}(Y)$, independent, such that $|X| = 2^\kappa$, where $\mathcal{P}(Y)$ denotes the power set of $Y$.*([H])

**Remark 5.2.18**

*(a)* Let $X$ be a strongly zero dimensional space where $sc(X) = \kappa$, i.e., there exists a separated cell $\mathfrak{U}$ with $|\mathfrak{U}| = \kappa$. By 5.2.17 there exists an independent subset, $\mathcal{I} \subseteq \mathcal{P}(\mathfrak{U})$ such that $|\mathcal{I}| = 2^\kappa$, say $\mathcal{I} = \{I_\alpha\}_{\alpha<2^\kappa}$. Then, for every $\alpha < \kappa$, we have $I_\alpha \subseteq \mathfrak{U}$; hence, by the fact that $\mathfrak{U}$ is a separated cell in $X$, there exists $f_\alpha \in C(X)$ such that $f[U] = 1$, for every $U \in I_\alpha$, and $f[U] = 0$, for every $U \in (\mathfrak{U} - I_\alpha)$. Now, $f_\alpha^{-1}(0), f_\alpha^{-1}(1) \in \mathcal{Z}[X]$, thus, by the strong zero dimensionality of $X$, there exists $C_\alpha \in \mathfrak{B}(X)$ where $C_\alpha \cap f_\alpha^{-1}(0) = \emptyset$, and $f_\alpha^{-1}(1) \subseteq C_\alpha$. Therefore,

$\mathfrak{I} = \{C_\alpha\}_{\alpha<2^\kappa}$ is an independent subset of $\mathfrak{B}(X)$. Indeed, for $F, G \in [\mathfrak{I}]^{<\omega}$ such that $F \cap G = \emptyset$,

$$(\wedge F) \wedge (\wedge G') \geq (\wedge_{i=1}^{n} f_{\alpha_i}^{-1}(1)) \wedge (\wedge_{j=1}^{m} f_{\delta_j}^{-1}(0))$$

$$\geq [\wedge_{i=1}^{n}(\vee(I_{\alpha_i}))] \wedge [\wedge_{j=1}^{m}(\vee((\mathfrak{U} - I_{\delta_j})))] \neq 0,$$

as $\mathcal{I}$ is independent in $\mathcal{P}(\mathfrak{U})$. Note that the operations $\wedge$, $\vee$ above are the set theoretic operations $\cap$, $\cup$, repectively, while $\geq$ denotes set containment, and 0 denotes the empyset. Thus, for a strongly zero dimensional space, $X$, $2^{sc(X)} \leq ind(X)$.

*(b)* A second method of constructing independent sets employs the cardinal function $dis(X)$, the *disconnectivity character* of $X$. This cardinal function is defined by $dis(X) =$

$$\sup\{\,\kappa\ :\ \text{for every}\ \mathfrak{O} \in [Coz[X]]^{<\kappa}, \text{we have}\ cl_X(\cup_{O\in\mathfrak{O}} O) \in \mathfrak{B}(X)\,\},$$

where $Coz[X]$ is the collection of all cozero sets of $X$, and $[Coz[X]]^{<\kappa}$ denotes the family of all subsets of $Coz[X]$ of cardinality less than $\kappa$. Utilizing the method of proof in 5.2.18(a), it can be shown that, if $X$ is a space such that $dis(X) \geq c(X)^+$, then $sc(X) = c(X)$, hence $2^{c(X)} \leq ind(X)$. An example where the preceeding inequality fails may be found by considering $\alpha(D(\kappa))$, the one point compactification of the discrete space of cardinality $\kappa > \aleph_0$. Then $ind(\alpha(D(\kappa))) = \aleph_0 < c(\alpha(D(\kappa))) = \kappa$.

In order to discuss the next method for construction of independent sets, a theorem of Shelah is needed. It addresses the existence of independent sets in a boolean algebra, $\mathfrak{B}$, where a particular relationship is satisfied beteen the cardinals $c(\mathcal{S}(\mathfrak{B}))$ and $w(\mathcal{S}(\mathfrak{B}))$. This relationship will presently be made precise.

**Definition 5.2.19** *Let $\mu$ and $\kappa$ be infinite cardinals. We say that $\mu$ is much less than $\kappa$, denoted $\mu \ll \kappa$, if, for every $\lambda < \kappa$, it is the case that $\mu^{<\lambda} < \kappa$, where $\mu^{<\lambda} = \sup\{\, \mu^\theta \,:\, \theta < \lambda \,\}$.*

The following may be found in [Ko], Chapter 10, and is due to Shelah.

**Theorem 5.2.20** *Let $\mathfrak{B}$ be a boolean algebra, and $\kappa$ an infinite cardinal, such that*

$$c(\mathcal{S}(\mathfrak{B}))^+ \ll \kappa.$$

*Then, for every $\mathfrak{D} \subseteq \mathfrak{B}$ of size $\kappa$, there exists an independent set, $\mathfrak{I} \subseteq \mathfrak{D}$, of size $\kappa$.*

**Remark 5.2.21** If $X$ is a $P$-space space such that $w(X) \geq ((c(X)^+)^+)$, assuming $GCH$, the generalized continuum hypothesis, it follows that $ind(X) = w(X)$. In fact, if $\kappa < w(X)$, then $(2^{c(X)})^{<\kappa} = (2^{c(X)})^\lambda$, where $\kappa = 2^\lambda$. Now, take $\kappa$ such that $w(X) = 2^\kappa$, and $\lambda$ such that $2^\lambda = \kappa$. Then, as $2^\lambda = \kappa \geq 2^{c(X)}$, hence $\lambda \geq c(X)$, we have $\lambda c(X) = \lambda$. It follows that

$$(2^{c(X)})^{<\kappa} = (2^{c(X)})^\lambda = 2^\lambda = \kappa < w(X).$$

Therefore, by 5.2.20, $ind(X) = w(X)$.

Our discussion of techniques for constructing independent sets concludes with a theorem of Šapirovskii. Again, a few preliminary definitions are required.

**Definition 5.2.22** *Let $X$ be a space and $x \in X$. A $\pi$-base for $x$ in $X$, is a collection, $\mathfrak{O}$, of open subsets of $X$, such that, for any open set $U$ with $x \in U$, there is an $O \in \mathfrak{O}$ with $O \subseteq U$. Note that it is not required that $x \in O$.*

**Definition 5.2.23** *Let $X$ be a space and $x \in X$. Define the $\pi$-character of $x$, dentoed $\pi\chi(x, X)$, by $\pi\chi(x, X) =$*

$$\inf\{\, \kappa \,:\, \text{there exists}, \mathfrak{O},\ a\ \pi\text{-}base\,for\,x \in X, \text{ where } |\mathfrak{O}| = \kappa \,\}.$$

**Definition 5.2.24** *Given a space, $X$, define*

$$\pi\chi_*(X) = \inf\{\, \pi\chi(x, X) \,:\, x \in X \,\}.$$

We now state the promised result; its proof may be found in [Ko], 10.16.

**Theorem 5.2.25** *Let $X$ be a space such that there exists an infinite, independent, subset of $\mathfrak{B}(X)$. Then*

$$ind(X) = \sup\{\, \pi\chi_*(C) \,:\, C \text{ a closed, subspace of } X \text{ with } |C| > 1 \,\}.$$

**Example 5.2.26** As an application of the above theorem, consider $D(\aleph_n)$, the discrete space of cardinality $\aleph_n$, where $\aleph_n > \mathfrak{c}$, and adjoin a point $\lambda$ to $D(\aleph_n)$. Topologize $D(\aleph_n) \cup \{\lambda\}$ so that all points in $D(\aleph_n)$ are open, and $U \in \mathcal{N}_\lambda$ if $|D \backslash U| \leq \aleph_0$, where $\mathcal{N}_\lambda$ denotes the collection of open sets containing $\lambda$; call this new space $\lambda D$. It is known, ([CN]), that $\beta(\lambda D)$, can be obtained by forming a quotient of $\beta D(\aleph_n)$ under the equivalence relation which identifies all $\omega_1$-uniform ultrafilters with $\mathcal{U}_\lambda$, the fixed ultrafilter at $\lambda$. (An ultrafilter is said to be $\omega_1$-uniform if it contains no set of cardinality less than $\aleph_1$.) Now, consider $ind(\lambda D)$. As $\lambda D$ is a P-space, we have $ind(\lambda D) \geq \mathfrak{c}$. Consequently, as $ind(\lambda D) = ind(\beta(\lambda D)$, by Šapirovskii's theorem it is enough to determine $\pi\chi_*(C)$, for every nontrivial, closed, subspace of $\beta(\lambda D)$. To this end, consider a point in $\beta(\lambda D) \backslash (\lambda D)$. For any such point $\mathcal{U}$, as $\mathcal{U}$ is by assumption not $\omega_1$-uniform, there exists $C \subseteq D$ with $|C| = \aleph_0$ such that $C \in \mathcal{U}$. Let $\{C_\beta\}_{\beta<\mathfrak{c}}$ be the collection of all infinite subsets of $C$. Then, for any $Z(f) \in \mathcal{U}$ we have $Z(f) \cap C \in \mathcal{U}$ and there exists a $\beta_0 < \mathfrak{c}$ with $C_{\beta_0} = Z(f) \cap C$. Hence $cl_{\beta(\lambda D)}(C_{\beta_0}) \subseteq cl_{\beta(\lambda D)}(Z(f))$, i.e. $\{cl_{\beta(\lambda D)}(C_\beta)\}_{\beta<\mathfrak{c}}$ is a base for $\mathcal{U}$ of cardinality less than or equal to $\mathfrak{c}$. Now, consider any $C$, a closed, nontrivial subset of $\beta(\lambda D)$. If $C \cap D \neq \emptyset$, then $\pi\chi_*(C) = 1$, as $D(\aleph_n)$ consists of isolated points. If $C \cap D = \emptyset$, then, by the assumption that $C$ is nontrivial, i.e., $|C| > 1$, there exists a non-$\omega_1$-uniform ultrafilter in $C$. Since, as

previuosly established, every such point, $x$, in $\beta(\lambda D)$ has a base of cardinality $\mathfrak{c}$ or less, it follows that $\pi\chi(x, C) \leq \mathfrak{c}$; consequently $\pi\chi_*(C) \leq \mathfrak{c}$. Therefore $ind(\lambda D) = \mathfrak{c}$.

**Definition 5.2.27** *Let $X$ be a space and $\kappa$ an infinite cardinal. Say that $X$ is a $\kappa$-Lindelöf space if every open cover of $X$ has a subcover of cardinality less than or equal to $\kappa$.*

**Definition 5.2.28** *For a space, $X$, the hereditary Lindelöf number of $X$, denoted $hL(X)$, is defined by*

$$hL(X) = \inf\{\,\kappa \,:\, \text{every subspace of } X \text{ is } \kappa\text{-}Lindel\ddot{o}f\,\}.$$

**Remark 5.2.29** This cardinal function provides an upper bound on the cardinality of generating sets of ideals of the boolean ring $\mathfrak{B}(X)$, for a compact, zero dimensional space, $X$. In fact, let $X$ be in the category $\mathcal{KZH}$, and $\mathfrak{I}$ an ideal in $\mathfrak{B}(X)$, i.e., $\mathfrak{I}$ is a subset of $\mathfrak{B}(X)$ which is closed under finite unions and subsets. Consider $\bigcup_{C\in\mathfrak{I}} C \subseteq X$. For $hL(X) = \kappa$, there exists $\mathfrak{I}' \subseteq \mathfrak{I}$ such that $\bigcup_{C\in\mathfrak{I}'} C = \bigcup_{C\in\mathfrak{I}} C$, with $|\mathfrak{I}'| = \kappa$. Now, as every $C \in \mathfrak{I}$ is clopen in $X$, hence compact, there exists $F_C \in [\mathfrak{I}']^{<\omega}$ such that $C \subseteq \bigcup_{C'\in F_C} C'$. Recalling that $\mathfrak{I}$ is an ideal in $\mathfrak{B}(X)$, we have $\mathfrak{I}$ is generated by $\mathfrak{I}'$, i.e., every ideal of $\mathfrak{B}(X)$ is generated by at most $\kappa$ many elements. Therefore, $hL(X) = \kappa$ implies that every ideal of $\mathfrak{B}(X)$ is generated by at most $\kappa$ many elements.

**Definition 5.2.30** *For a space, $X$, define the depth of $X$, denoted by $dep(X)$, by $dep(X)=$*

$$\sup\{\,\kappa \,:\, \text{there exists } \mathcal{C} \subseteq \mathfrak{B}(X), \text{ well ordered under } \subseteq \text{ with } |\mathcal{C}| = \kappa\,\}.$$

**Definition 5.2.31** *Let $X$ be a space and define the hereditarily closed depth of $X$, denoted $hcdep(X)$, by*

$$hcdep(X) = \sup\{\,\kappa \,:\, dep(C) = \kappa \text{ where } C \subseteq X,\ C \text{ closed}\,\}.$$

**Definition 5.2.32** *Given a space, X, and $x \in X$, define the tightness of x in X, denoted t(x,X), by t(x,X)=*

$$\inf\{\,\kappa\,:\,\text{if}\, S \subseteq X \,\text{and}\, x \in cl(S),\,\text{then}\, \exists\; T \subseteq S,\,\text{with}\, x \in cl(T)\,\text{and}\, |T| \leq \kappa\,\}.$$

**Definition 5.2.33** *Let X be a space, and define the tightness of X, denoted t(X), by*

$$t(X) = \inf\{\,\kappa\,:\, t(x,X) \leq \kappa,\,\text{for every}\, x \in X\,\}.$$

**Remark 5.2.34** It is known, [M], that $t(X) = hcdep(X)$. This fact will play a crucial role in the main theorem of Chapter 7.

CHAPTER 6
Bounding the Global Dimension of $C(X)$

Having developed the necessary equipment regarding cardinal functions on Stone and $P$-spaces, we are now in a position to compute upper and lower bounds for the global dimension of $C(X)$. This shall be achieved by way of a two step process. First, in section one, the cardinal functions developed in Chapter 5 will be applied to the boolean ring $\mathfrak{B}(X)$, for $X$ a $P$-space, or in more generality, to spaces $X$ in $\mathcal{KZH}$. These applications, when viewed in the light of the results in Chapter 3 by Osofsky, will supply upper and lower bounds for the global dimension of $\mathfrak{B}(X)$. Then, in the second section of this chapter, analogues of the results regarding boolean rings will developed for the ring $C(X)$, when $X$ is a $P$-space. Finally, in the third section we will consider the problem of detemining what restrictions are forced on spaces, $X$, when the global dimension of $C(X)$ is two.

## 6.1 Global Dimension of $\mathfrak{B}(X)$

Of the results cited in Chapter 3, Section 1, there are two which will be invoked repeatedly; thus, we remind the reader of them here.

**Theorem 6.1.1** [Osl]: *Let $A$ be a ring so that every ideal of $A$ is generated by at most $\aleph_n$ elements. Then*

$$gD(A) \leq wD(A) + n + 1.$$

**Definition 6.1.2** *Let $\{e_\gamma\}_{\gamma<\alpha}$ be a collection of idempotents where $\alpha$ is an ordinal. We say the collection $\{e_\gamma\}_{\gamma<\alpha}$ is independent if*

$$(\Pi_{k\leq n}e_{\gamma_k})\cdot(\Pi_{j\leq m}(1-e_{\delta_j}))\neq 0,$$

*whenever $\{e_{\gamma_k}\}_{k\leq n}\cap\{e_{\delta_j}\}_{j\leq m}=\emptyset$, where $m,n<\omega$.*

**Theorem 6.1.3** *Let $\{e_\gamma\}_{\gamma<\alpha}$ be a collection of independent idempotents of $A$ and let $I=\Sigma_{\gamma<\alpha}e_\gamma A$. If there exists an $n\in\omega$ such that no ordinal of cardinality $<\aleph_n$ is cofinal in $\alpha$ then $pd(I)\geq n$.*

A technical result stated in Chapter 2, which will be called upon repeatedely, is the following:

**Proposition 6.1.4** *Let $A$ be a ring where $gD(A)\neq 0$. Then*

$$gD(A)=\sup\{\,pd(I)\,:\,I\,\text{an ideal of}\,A\,\}+1.$$

**Remark 6.1.5** To be able to use the reuslts in 6.1.1 and 6.1.3, we must first note the connection between the set theoretic operations of the boolean algebra $\mathfrak{B}(X)$, and the multiplication and order on the collection $\{e_B\}_{B\in\mathfrak{B}(X)}$ as a subset of the ring $C(X)$. Although this distinction may seem only to be a semantic one, it still merits a pause for explaination. Note that for every $B\in\mathfrak{B}(X)$, the characteristic function on B, which we will denote by $e_B$, is in fact a continuous function on $X$, i.e., $e_B\in C(X)$. Considering the pointwise order and multiplication on $C(X)$, for $B_1,B_2\in\mathfrak{B}(X)$, observe the following equalities:

*(a)* $e_{B_1}\vee e_{B_2}=e_{(B_1\cup B_2)}$;

*(b)* $e_{B_1}\cdot e_{B_2}=e_{B_1}\wedge e_{B_2}=e_{(B_1\cap B_2)}$;

*(c)* $(1 - e_{B_1}) = e_{(X-B_1)}$;

where 1 denotes the constant identity element of $C(X)$. As a consequence of the preceeding, by 5.2.15 and 6.1.2, if $\mathfrak{I}$ is an independent subset of $\mathfrak{B}(X)$, then $\{e_B\}_{B\in\mathfrak{I}}$ is an independent set of idempotents in the boolean subring $\mathfrak{B}(X)$ of $C(X)$.

A direct application of the above remark and 6.1.1 and 6.1.3 yield the following two theorems:

**Theorem 6.1.6** *Let $X$ be a space such that ind(X)=$\aleph_n$. Then*

$$gD(\mathfrak{B}(X)) \geq n+1.$$

**Proof:** By 6.1.5, since $\mathrm{ind}(\mathfrak{B}(X)) = \aleph_n$, there exists an independent set of idempotents, $\mathcal{E} = \{e_B\}_{B\in\mathfrak{B}(X)}$, in the boolean ring $\mathfrak{B}(X)$, where $|\mathcal{E}| = \aleph_n$. As no ordinal strictly less than $\aleph_n$ is cofinal in $\aleph_n$, by 6.1.3,

$$pd(\sum_{e_B\in\mathcal{E}}(e_B\mathfrak{B}(X))) \geq n.$$

Invoking 6.1.4, we obtain $gD(\mathfrak{B}(X)) \geq n+1$. □

**Theorem 6.1.7** *Let $X$ be in $\mathcal{KZH}$ such that hL(X)=$\aleph_n$. Then*

$$gD(\mathfrak{B}(X)) \leq n+1.$$

**Proof:** By 5.2.29, every ideal of the boolean ring, $\mathfrak{B}(X)$, is generated by at most $\aleph_n$ many elements. Hence, employing 6.1.1, we obtain $gD(\mathfrak{B}(X)) \leq wD(\mathfrak{B}(X)) + n + 1$. Now, as $\mathfrak{B}(X)$ is boolean, hence von Neumann regular, it follows that $wD(\mathfrak{B}(X)) = 0$; consequently, $gD(\mathfrak{B}(X)) \leq n+1$. □

**Remark 6.1.8** A curious proof of a known topological fact follows from 6.1.6 and 6.1.7. Indeed, if $X$ is in $\mathcal{KZH}$ and $ind(X) = \aleph_m$, and $hL(X) = \aleph_n$, then $m+1 \leq gD(\mathfrak{B}(X)) \leq n+1$; hence we have shown $ind(X) \leq hL(X)$, for $X$ in $\mathcal{KZH}$.

**Corollary 6.1.9** *Let $X$ be a $P$-space such that $hL(\beta X) = \aleph_n$. Then*

$$gD(\mathfrak{B}(X)) \leq n + 1.$$

**Proof:** From 5.2.29 $\mathfrak{B}(X) \cong \mathfrak{B}(\beta X)$; thus, every ideal of $\mathfrak{B}(X)$ is generated by at most $\aleph_n$ many elements, and the result follows. □

At this point we have proved that, for a $P$-space, $X$, the cardinal functions $ind(\cdot)$ and $hL(\cdot)$ provide lower and upper bounds, respectively, for the global dimension of $\mathfrak{B}(X)$. For two particular classes of spaces we can do better than this and make definitive statements about the global dimension of $\mathfrak{B}(X)$ for these classes.

**Theorem 6.1.10** *Let $X$ be an extremally disconnected space; i.e., suppose that the closure of every open set is again open. If $X$ is in $\mathcal{KZH}$, then*

$$gD(\mathfrak{B}(X)) = n + 1,$$

*where $w(X) = \aleph_n$.*

**Proof:** It is a well known fact ([Ki]) that if $X$ is extremally disconnected, then $ind(X) = w(X)$. Since $hL(X) \leq w(X)$, this implies $ind(X) = hL(X)$; then the result follows from 6.1.6 and 6.1.7. □

**Theorem 6.1.11** *Let $X$ be in $\mathcal{KZH}$ such that $c(X)^+ \ll hL(X)$. Then*

$$gD(\mathfrak{B}(X)) = n + 1,$$

*where $ind(X) = hL(X) = \aleph_n$.*

**Proof:** By 5.2.20, since $c(\mathfrak{B}(X))^+ \ll hL(X)$, there exists $\mathfrak{I}$, independent in $\mathfrak{B}(X)$ with $|\mathfrak{I}| = hL(X)$. Therefore, $ind(X) = hL(X)$, and the result follows from 6.1.6 and 6.1.7. □

**Remark 6.1.12** From the above results one might be tempted to conjecture that, for $X$ in $\mathcal{KZH}$, $gD(\mathfrak{B}(X)) = n+1$, where $ind(X) = \aleph_n$. However, an example which witnesses the failure of this conjecture will be provided in Chapter 7.

## 6.2 Global Dimension of $C(X)$

We now consider the problem of finding upper and lower bounds for the global dimension of $C(X)$. Many of the results in this section follow from the boolean case treated in Section 1 of this chapter, although the first results on the upper bounds of $C(X)$ do not, and may even be applied to the class of $F$-spaces, which is more general than the class of $P$-spaces.

Applying 6.1.1 to the ring $C(X)$, the following is immediate:

**Proposition 6.2.1** *Let $X$ be a space such that every ideal of $C(X)$ is generated by at most $\aleph_n$ many elements. Then*

$$gD(C(X)) \leq wD(C(X)) + n + 1.$$

Interpreting this result for $F$-spaces and $P$-spaces yields:

**Corollary 6.2.2** *Let $X$ be an $F$-space such that every ideal of $C(X)$ is generated by at most $\aleph_n$ many elements. Then*

$$gD(C(X)) \leq n + 2.$$

**Proof:** If $X$ is an $F$-space, then by 4.1.5, $wD(C(X)) \leq 1$; now apply 6.2.1. □

**Corollary 6.2.3** *Let $X$ be a $P$-space such that every ideal of $C(X)$ is generated by at most $\aleph_n$ many elements. Then*

$$gD(C(X)) \leq n + 1.$$

**Proof:** Immediate by 6.2.1 and the fact that $wD(C(X)) = 0$, since, by 5.1.2, $C(X)$ is von Neumann regular. □

**Remark 6.2.4** Note that 6.2.2 and 6.2.3 allow us to obtain upper bounds for the global dimension of $C(X)$, for $X$ an $F$-space or $P$-space, in terms of the density character of the space $X$. In fact, if $Y$ is a dense subspace of $X$, then there exists a ring embedding $\phi : C(X) \longrightarrow C(Y)$. Then, since every $f \in C(X)$ is determined by its values on $Y$, $|C(X)| \leq |C(Y)| \leq 2^{|Y|} = 2^{d(X)}$, where $2^{d(X)} = \aleph_n$ for some $n \in \mathbb{N}$. As the cardinality of the ring $C(X)$ is bounded by $\aleph_n$, certainly every ideal of $C(X)$ is generated by at most $\aleph_n$ many elements. From this observation we conclude:

**Corollary 6.2.5** *Let $X$ be an $F$-space such that $2^{d(X)} = \aleph_n$. Then*

$$gD(C(X)) \leq n + 2.$$

**Corollary 6.2.6** *Let $X$ be a $P$-space such that $2^{d(X)} = \aleph_n$. Then*

$$gD(C(X)) \leq n + 1.$$

In order to be able to improve the rather coarse upper bounds stated in 6.2.5 and 6.2.6, we will turn to the topological results obtained in Chapter 5. As these results rely on the assumption that, for a given space $X$, $\mathfrak{B}(X)$ is nontrivial, we are forced to leave behind the possibility of directly investigating $F$-spaces. For, recall that in the remarks following 4.1.1, it was noted that there exist $F$-spaces which are in fact connected. Hence, our discussion of the global dimension of $C(X)$ will be primarily restricted to the class of $P$-spaces. We begin by applying the results from Section 1 of this chapter.

**Proposition 6.2.7** *Let $X$ be a space such that $ind(X) = \aleph_n$. Then*

$$gD(C(X)) \geq n + 1.$$

**Proof:** Completely analogous to 6.1.6. □

**Proposition 6.2.8** *Let $X$ be an infinite $P$-space and $\mathfrak{c} = \aleph_n$. Then*

$$gD(C(X)) \geq n+1.$$

**Proof:** We claim that every infinite $P$-space, $X$, has a separated cell of size $\aleph_0$. In fact, as $X$ is Hausdorff, it is elementary to show $c(X) \geq \aleph_0$; thus $X$ has a collection of pairwise disjiont clopen sets. Recalling, ([GJ]), that $\mathcal{Z}[X] = \mathfrak{B}(X)$ is closed under countable intersections, it follows that $\mathcal{Z}[X]$ is closed under countable unions as well. Therefore, $sc(X) \geq \aleph_0$, and so $ind(X) \geq 2^{sc(X)} \geq 2^{\aleph_0} = \mathfrak{c}$. The result now follows from 6.2.7. □

**Proposition 6.2.9** *Let $X$ be a $P$-space, where $hL(\beta X) = \aleph_n$. Then*

$$gD(C(X)) \leq n+1.$$

**Proof:** As $X$ is a $P$-space, every ideal of $C(X)$ is generated by idempotents, hence by the characteristic functions of elements of $\mathfrak{B}(X)$. Then, by 5.2.29, every ideal of $C(X)$ is generated by a collection of idempotents, $\mathcal{E}$, where $|\mathcal{E}|$ is at most $\aleph_n$. Now apply 6.2.3. □

**Theorem 6.2.10** *Let $X$ be a space such that $dis(X) \geq c(X)^+$. Then,*

$$gD(C(X)) \geq n+1,$$

*where $2^{c(X)} = \aleph_n$. If $2^{c(X)} > \aleph_n$ for every $n \in \mathbb{N}$, then $gD(C(X)) = \infty$.*

**Proof:** By 5.2.18(b), if $dis(X) \geq c(X)^+$, then $ind(X) \geq 2^{c(X)}$. Hence, if $2^{c(X)} = \aleph_n$, then $ind(X) \geq \aleph_n$, and 6.2.7 yields $gD(C(X)) \geq n+1$. If $2^{c(X)} > \aleph_n$ for every $n \in \mathbb{N}$, then for every $n$, by 6.2.7, there exists an ideal in $C(X)$ such that $pd(I) \geq n$. Invoking 6.1.4, we conclude $gD(C(X)) = \infty$. □

**Theorem 6.2.11** *Let $X$ be a strongly zero dimensional space such that $2^{sc(X)} = \aleph_n$. Then*

$$gD(C(X)) \geq n + 1.$$

**Proof:** By 5.2.18(a), $\aleph_n = 2^{sc(X)} \leq ind(X)$. Again, there exists an independent collection, $\mathcal{E}$, of idempotents in the ring $C(X)$, where $|\mathcal{E}| = \aleph_n$. By 6.2.7, the result follows. □

**Theorem 6.2.12** (GCH) *Let $X$ be a $P$-space such that $w(\beta X) \geq (c(\beta X)^+)^+$. Then*

$$gD(C(X)) = n + 1,$$

*where $w(\beta X) = \aleph_n$.*

**Proof:** By 5.2.21, if $w(\beta X) \geq (c(\beta X)^+)^+$, then $ind(\beta X) = w(\beta X) = \aleph_n$. Since $X$ is a $P$-space, $ind(X) = ind(\beta X)$; hence $ind(X) = \aleph_n$. Next, as

$$\aleph_n = w(\beta X) \geq hL(\beta X) \geq ind(\beta X) = \aleph_n,$$

we have $ind(X) = hL(\beta X) = \aleph_n$. It follows from 6.2.9 and 6.2.7, that $gD(C(X)) = n + 1$. □

We shall now employ the above results to compute $gD(C(X))$ for a number of $P$-spaces $X$.

**Example 6.2.13** Consider the discrete space $D(\aleph_0)$ and $C(D(\aleph_0))$, the ring of all real valued sequences. Then, as $|C(D(\aleph_0)| = \mathfrak{c}$, by 6.2.3 we have $gD(C(D(\aleph_0))) \leq k + 1$ where $\mathfrak{c} = \aleph_k$. Thus, by 6.2.8, we obtain $gD(C(D(\aleph_0))) = k + 1$. We note that this fact was proved by Osofsky in [Os1].

**Example 6.2.14** For $\lambda D(\mathfrak{c})$ as discussed in 5.2.10, consider $C(\lambda D(\mathfrak{c}))$. This is the ring of all real valued functions on $D$ which are constant on all but a countable subset of $D$. Since $|C(\lambda D(\mathfrak{c}))| = \mathfrak{c}$, as above, we obtain $gD(C(\lambda D(\mathfrak{c}))) \leq k+1$ where $\mathfrak{c}=\aleph_k$. In addition, again, by 6.2.8, we have $ind(\lambda D(\mathfrak{c})) \geq \mathfrak{c}$. It then follows that $gD(C(\lambda D(\mathfrak{c}))) = k+1$ where $\mathfrak{c} = \aleph_k$.

**Example 6.2.15** Recall that an $\eta_1$-set is a totally ordered set, $\mathfrak{Y}$, such that, for any $A$,$B \subseteq \mathfrak{Y}$ with $A < B$ and $|A|$,$|B| \leq \aleph_0$, there exists an $x \in \mathfrak{Y}$ where $A < x < B$. Now, if $\mathfrak{Y}$ is an $\eta_1$-set of cardinality $\mathfrak{c}$, then, endowing it with the interval topology, it becomes a $P$-space, ([GJ], 13P.1). Using arguments identical to those in 5.2.13 and 5.2.14, we obtain $gD(C(\mathfrak{Y}))=k+1$ where $\mathfrak{c}=\aleph_k$.

**Remark 6.2.16** As the separated cellularity of the spaces in all of the above examples is $\omega$, one might be tempted to conjecture that $gD(C(X))=k+1$, where $2^{sc(X)}=\aleph_k$, for any P-space $X$. However, the following shows that this is false.

**Example 6.2.17** We shall consider $D^{(2^{\mathfrak{c}})^+}$. Now apply the $P$-space coreflection to this space to obtain the $P$-space $(D^{(2^{\mathfrak{c}})^+})_\delta$ . Next, consider the subspace of this space consisting of all points which have at most a countable number of nonzero entries; call this subspace $X_\delta$. It can be shown that $c(X_\delta)=\mathfrak{c}$, hence $sc(X_\delta) \leq \mathfrak{c}$. Moreover, by 5.2.20, we have $ind(X_\delta)=(2^{\mathfrak{c}})^+= |\mathfrak{B}(X_\delta)|$; consequently, $gD(C(X_\delta))=k+1$, where $(2^{\mathfrak{c}})^+=\aleph_k > 2^{\mathfrak{c}} \geq 2^{sc(X_\delta)}$. We mention that this space was suggested by O. Alas.

We conclude this list of examples with one which shows that the global dimension of a P-space need not be finite.

**Example 6.2.18** Consider the space $\lambda D(\aleph_{\omega_1})$; denote this space by $X$. Then $X$ is a P-space for which $sc(X) > \aleph_n$, for every $n \in \mathbb{N}$; consequently, by 6.2.7 and 6.1.4, we have $gD(C(X))=\infty$.

## 6.3 When the Global Dimension of $C(X)$ is 2

Considering the examples presented above, one sees that in many cases the global dimension of $C(X)$ when $X$ is a $P$-space takes on the value $n+1$, where $\mathfrak{c} = \aleph_n$. Thus, if one assumes $CH$, i.e., $\mathfrak{c} = \aleph_1$, then the global dimension of the above mentioned $P$-spaces is two. This naturally gives rise to the problem of determining what restrictions, if any, are forced upon a space $X$, when the global dimension of $C(X)$ is two. In this section we detemine one such restriction, even in the absence of set theoretic assumptions such as $CH$.

The following preliminary results can be found in [Gl], 4.2.

**Theorem 6.3.1** *Let $A$ be a commutative ring with identity. Then, for every $P \in$ Spec$(A)$, $gD(A) \geq gD(A_P)$.*

**Theorem 6.3.2** *Let $A$ be a commutative ring with identity. If $gD(A) = 2$, the $A_P$ is a domain for every $P \in$ Spec$(A)$.*

**Definition 6.3.3** *Let $A$ be a totally ordered valuation domain with bounded inversion , and let $\mathfrak{m}$ be the maximal ideal of $A$. Call a subset $\mathcal{C} = \{a_\lambda\}_{\lambda<\kappa}$ of positive elements of $A$, which is totally ordered under the $f$-ring order on $A$, a canonical chain in $\mathfrak{m}$, if for every $\lambda < \kappa$, $(a_\lambda)$, the principal ideal generated by $a_\lambda$, is strictly contained in $(a_{\lambda+1})$. We call the cardinal, $\kappa$, the length of the chain $\mathcal{C}$, and denote it by $l(\mathcal{C})$.*

**Definition 6.3.4** *Again, let $A$ be a totally ordered valuation domain with bounded inversion, and $\mathfrak{m}$ the maximal ideal of $A$. Define the length of $A$, denoted $l(A)$, by*

$$l(A) = \sup\{\kappa : l(\mathcal{C}) = \kappa, \ \mathcal{C}\,\text{a canonical chain in } \mathfrak{m}\}.$$

*Then define the index of $A$, denoted $i(A)$, by $i(A) = n$ if $l(A) = \aleph_n$.*

**Theorem 6.3.5** *Let $A$ be totally ordered valuation domain with bounded inversion, such that $A$ is not a field. If $i(A)$ is finite,*

$$gD(A) = i(A) + 2.$$

**Proof:** Note that $A$ is 1-convex. Hence, the inclusion ordering on the principal ideals generated by the elements of the canonical chain is compatable with the ordering of the canonical chain. Now apply 3.1.12. □

To obtain the desired result, we apply the above machinery to the localizations of the ring $C(X)$. First, a point version of the $P$-space property is defined.

**Definition 6.3.6** *Let $X$ be a space and $x \in X$. Say that $x$ is a $P$-point if $\mathfrak{m}_x = \mathcal{O}(\mathfrak{m}_x)$.*

**Proposition 6.3.7** *Let $X$ be a space and $x \in X$, $x$ not a $P$-point, such that $C(X)_{\mathfrak{m}_x}$ is a valuation domain. Then*

$$gD(C(X)_{\mathfrak{m}_x}) \geq 3.$$

**Proof:** It is shown in [MW], that $C(X)_{\mathfrak{m}_x} \cong C(X)/\mathcal{O}(\mathfrak{m}_x)$, which is not a field, as, by assumption, $x$ is not a $P$-point. Therefore, by [GJ], 14.19, $C(X)/\mathcal{O}(\mathfrak{m}_x)$ contains an $\eta_1$-set, $\mathfrak{Y}$. Now choose a cofinal chain, $\mathcal{C}$ in $\mathfrak{Y}$; say $\mathcal{C} = \{a_\lambda\}_{\lambda < \aleph_1}$. Then, as $\mathcal{C}$ is cofinal in an $\eta_1$-set, $\mathcal{C}$ forms a canonical chain in $\mathfrak{m}_x$ where $l(\mathcal{C}) = \aleph_1$; thus, $1 \leq i(C(X)/\mathcal{O}(\mathfrak{m}))$. By 6.3.5 we conclude that $gD(C(X)_{\mathfrak{m}_x}) \geq 3$. □

**Theorem 6.3.8** *Let $X$ be a space such that $gD(C(X)) = 2$. Then $X$ is a $P$-space.*

**Proof:** If $gD(C(X)) = 2$, then by 6.3.2 we have $C(X)_P$ is a domain for every $P \in Spec(C(X))$. In particular, $C(X)_{\mathfrak{m}_x} \cong C(X)/\mathcal{O}(\mathfrak{m}_x)$ is a domain for every $\mathfrak{m}_x \in Max(C(X))$. By [GJ], Chapter 14, we have $C(X)_{\mathfrak{m}_x}$ is a valuation domain for every $\mathfrak{m}_x \in Max(C(X))$. Invoking 6.3.1, $gD(C(X)_{\mathfrak{m}_x}) \leq gD(C(X)) = 2$; consequently, by 6.3.7, every $x \in X$ must be a $P$-point. Therefore, utilizing 5.1.2, $X$ is a $P$-space. □

# CHAPTER 7
# A New Lower Bound for Global Dimension of a von Neumann $f$-Ring

The results concerning the global dimension of $C(X)$ in the previous chapter suggest that this invariant might be determined by the independence character of the space $X$. The remarkable work by Osofsky on the global dimension of rings with idempotents confronted this issue and partially answered this conjecture by showing that the global dimension of a ring may be bounded from below by the cardinality of *almost independent* sets of idempotents of the ring, ([Os2]). It is in this chapter that an alternative to this lower bound is offered. We prove here, that for a von Neumann $f$-ring, $A$, the global dimension of $A$ is bounded below by the tightness of the maximal ideal space of $A$. Thus, the problem of constructing von Neumann rings of continuous functions for which the global dimension is strictly greater than the lower bound provided by the independence character of the space $X$, is reduced to the construction of $P$-spaces, $X$, such that $t(\beta X) > ind(\beta X)$.

## 7.1 The Resolution

The road to our result begins with a construction of a projective resolution for an ideal generated by a linearly ordered set of idempotents of a von Neumann $f$-ring. We note that the construction which follows is a minor variant of the construction by Osofsky outlined in 3.1.13.

**Definition 7.1.1** *Let $A$ be a von Neumann $f$-ring, and $\mathcal{E}$ a collection of idempotents of $A$ linearly ordered by the $f$-ring order on $A$, with no greatest element. We definie a projective resolution of $I_{\mathcal{E}}$, the ideal generated by $\mathcal{E}$, as follows:*

*(a)* $P_{-1}(\mathcal{E}) = I_{\mathcal{E}}$;

*(b)* *For* $n \geq 0$ *let*

$$P_n(\mathcal{E}) = \bigoplus_{e_0 > \ldots > e_n} (A < e_0, \ldots, e_n >)$$

*where* $\{e_0 > \ldots > e_n\}$ *ranges over all subsets of* $\mathcal{E}$ *of size* $n+1$ *satisfying the given decreasing order condition. Note then that* $P_n(\mathcal{E})$ *is actually a free A-module.*

*(c)* *For* $n \geq 1$ *define* $d_n : P_n(\mathcal{E}) \longrightarrow P_{n-1}(\mathcal{E})$ *by* $d_n(< e_0, \ldots, e_n >) =$

$$[\sum_{0=i}^{n-1}(-1)^i < e_0, \ldots, \hat{e}_i, \ldots, e_n >] + (-1)^n e_n < e_o, \ldots, e_{n-1} >$$

*where* $< e_0, \ldots, \hat{e}_i, \ldots, e_n >$ *is* $< e_0, \ldots, e_n >$ *with* $e_i$ *deleted.*

*(d)* *For* $n = 0$ *define* $d_0 : P_0(\mathcal{E}) \longrightarrow P_{-1}(\mathcal{E})$ *by* $d_0(ae_0) = ae_0$

*(e)* *Given* $e \in \mathcal{E}$ *and* $n \geq 0$, *let* $e^* : P_n(s(e)) \longrightarrow P_{n+1}(\overline{s}(e))$ *be defined by* $e^*(< e_0, \ldots, e_n >) =< e, e_0, \ldots, e_n >$, *where* $s(e) = \{f \in \mathcal{E} : f < e\}$ *and* $\overline{s}(e) = \{f \in \mathcal{E} : f \leq e\}$.

*(f)* *Given* $e \in \mathcal{E}$ *and* $n = -1$ *define* $e^* : P_{-1}(\overline{s}(e)) \longrightarrow P_0(\overline{s}(e))$ *by* $e^*(ae) = a < e >$.

**Theorem 7.1.2** *The sequence*

$$\mathcal{P}(\mathcal{E}) : \cdots \longrightarrow P_n(\mathcal{E}) \xrightarrow{d_n} P_{n-1}(\mathcal{E}) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0(\mathcal{E}) \xrightarrow{d_0} I \longrightarrow 0$$

*is a projective resolution of* $I_{\mathcal{E}}$.

**Proof:** The proof will proceed through three stages.

(a) First, note that, if $a \in P_n(\overline{s}(e))$, then $d_{n+1}(e^*(a)) = a - e^*(d_n(a))$. In fact, if $n \geq 1$, for $a =< e_0, \ldots, e_n >$, we have

$$d_{n+1}(e^*(< e_0, \ldots, e_n >) = d_{n+1}(< e, e_0, \ldots, e_n >) =< e_0, \ldots, e_n > +$$

$$\sum_{i=0}^{n-1}(-1)^{i+1} < e, ..., \hat{e}_i, ..., e_n > +(-1)^{n+1} e_n < e, ..., e_{n-1} >=$$

$$< e_0, ..., e_n > -(\sum_{i=0}^{n-1}(-1)^i < e, ..., \hat{e}_i, ..., e_n > +(-1)^n e_n < e, ..., e_{n-1} >)$$

$$= a - e^*([\sum_{i=0}^{n-1}(-1)^i < e_0, ..., \hat{e}_i, ..., e_n >] + (-1)^n e_n < e_0, ..., e_{n-1} >)$$

$$= a - e^*(d_n(a)).$$

For the case $n = 0$, we have

$$d_1(e^*(< e_0 >)) = d_1(< e, e_0 >) =< e_0 > -e_0 < e >$$

$$=< e_0 > -e^*(e_0 e) =< e_0 > -e^*(d_0(e_0)).$$

Then, extending by linearity in both cases, the result is verified.

(b) The next step is to show that $\mathcal{P}(\mathcal{E})$ is a complex, i.e., for every $n \in \mathbb{N}$, $d_{n-1}d_n = 0$. As in (a), this will be proved for elements of the form $< e_0, ..., e_n >$, and then extention by linearity will complete the result. Thus, consider $< e_o, ..., e_n >\in P_n$. Then

$$d_{n+1}(d_n(< e_0, ..., e_n >)) =$$

$$d_{n+1}([\sum_{i=0}^{n-1}(-1)^i < e_0, ..., \hat{e}_i, ..., e_n >] + (-1)^n e_n < e_0, ..., e_{n-1} >) =$$

$$[\sum_{i=0}^{n-1}(-1)^i d_{n+1}(< e_0, ..., \hat{e}_i, ..., e_n >)] + (-1)^n e_n d_{n+1}(< e_0, ..., e_{n-1} >) =$$

$$\sum_{i=0}^{n-1}(\sum_{j=0}^{n-2}(-1)^{i+j} < e_0, ..., \hat{e}_k, ..., \hat{e}_{k'}, ..., e_n >)+$$

$$\sum_{i=0}^{n-1}(-1)^{i+n-1} e_n < e_0, ..., \hat{e}_i, ..., e_{n-1} > +$$

$$[\sum_{j=0}^{n-2}(-1)^{j+n} e_n < e_0, ..., \hat{e}_j, ..., e_{n-1} >] + (-1)^{2n-1} e_n e_{n-1} < e_0, ..., e_{n-2} >$$

$$= \sum_{i=0}^{n-1}(-1)^{i+n-1} e_n < e_0, ..., \hat{e}_i, ..., e_{n-1} > +$$

$$\sum_{j=0}^{n-1}(-1)^{j+n} e_n < e_0, ..., \hat{e}_j, ..., e_{n-1} >= 0,$$

as $e_n e_{n-1} = e_n$, and where $k' = j$ if $k = i$, or $k' = i$ if $k = j$. Therefore, extending by linearity, $\mathcal{P}(\mathcal{E})$ is a complex.

(c) The final stage of the proof is to show that $\mathcal{P}(\mathcal{E})$ is an exact sequence. Thus, we must prove $ker(d_n) = Im(d_{n+1})$, for every $n$. However, this follows easily from the identity in (a). In fact, let $p \in ker(d_n)$, and choose $e \in \mathcal{E}$ such that $e > e_i$, for every $e_i$ appearing in a basis element used to write $p$. Then, by (a), $d_{n+1}(e^*(p)) = p - e^*(d_n(p)) = p$, as $p \in ker(d_n)$, i.e., $p \in Im(d_{n+1})$.

Combining (a), (b), and (c), the theorem is proved. □

At this stage a technical result is proved which is, again, a minor variation of a theorem of Osofsky.([Os1])

**Theorem 7.1.3** *Let $\mathcal{E}$, $P_k(\mathcal{E})$, and $d_k$, be as in 7.1.1. In addition, assume $|\mathcal{E}| = \aleph_n$, no set of cardinality $< \aleph_n$ generates $I_{\mathcal{E}}$, and $pd(I_{\mathcal{E}}) \leq k < \infty$. Then, for any $\hat{\mathcal{F}} \subseteq \mathcal{E}$, of cardinality $\aleph_k$, there exists $\mathcal{F} \subseteq \mathcal{E}$ such that:*

*(a) $|\mathcal{F}| = \aleph_k$;*

*(b) no subset of cardinality strictly less than $\aleph_k$ generates $I_{\mathcal{F}}$;*

*(c) $d_k(P_k(\mathcal{F}))$ is a summand of $d_k(P_k(\mathcal{E}))$.*

**Proof:** Since $pd(I_{\mathcal{E}}) \leq k$, we have $d_k(P_k(\mathcal{E}))$ is projective; hence, there exists an $A$-module, M, such that

$$d_k(P_k(\mathcal{E})) \bigoplus M = \bigoplus_{\gamma \in \Gamma} A_\gamma,$$

for some $\Gamma$. Next, let $\hat{\mathcal{F}}$ be any subset of $\mathcal{E}$ with cardinality $\aleph_k$. Consider any $G \in [\hat{\mathcal{F}}]^{<\omega}$, and denote by $s(G)$ an element of $\mathcal{E}$ strictly larger than every element of $G$. Now, if $|G| \leq k$, then leave $G$ alone. If $|G| > k$, then consider $d_k(P_k(G^*))$, where $G^* = G \cup \{s(G)\}$. Next, let $b \in d_k(P_k(G^*))$. Note that $b$ is also in the free module $\bigoplus_{\gamma \in \Gamma} A_\gamma$. Consider the set $F_b \in [\Gamma]^{<\omega}$ consisting of all the basis elements of the free module which are used in expressing $b$, i.e., the elements of $F_b$ are of the form $\gamma_i$ where $b = \sum a_i x_{\gamma_i}$. Now, for each $\gamma_i \in F_b$, let $B_{b,\gamma_i} \subseteq \mathcal{E}$ consist of all $e \in \mathcal{E}$ which appear in basis elements of $P_{k-1}(\mathcal{E})$ used in expressing $x_{\gamma_i} \in d_k(P_k(\mathcal{E}))$. Then define $H_b = \bigcup_{\gamma_i \in F_b} B_{b,\gamma_i}$ and $G' = \bigcup_{b \in d_k(P_k(G^*))} H_b$. This defines a function

$$(\cdot)' : [[\mathcal{E}]^{<\omega}]^{<\omega} \longrightarrow [\mathcal{E}]^{<\omega}.$$

By [Os1], 2.41, there exists a set $Y \subseteq [\mathcal{E}]^{<\omega}$, such that $Y \supseteq \hat{\mathcal{F}}$, $|Y| = \aleph_k$, and no subset of cardinality $< \aleph_k$ is cofinal in $Y$. Thus, defining $\mathcal{F} = \bigcup_{y \in Y} y$, it follows that $\mathcal{F}$ satisfies (a) and (b) of the theorem. As for (c), define $\Gamma_0 =$

$$\{\gamma \in \Gamma \ : \ \exists\, d \in d_k(P_k(\mathcal{F})), \text{ such that } d = \sum_{j=1}^{m} r_j x_{\gamma_j}, \text{ and } \exists\, \gamma_{j_0} \text{ with } \gamma = \gamma_{j_0}, \}$$

Then, defining $\Gamma_1 = (\Gamma - \Gamma_0)$, by the definition of $(\cdot)'$, we have

$$d_k(P_k(\mathcal{E})) = (d_k(P_k(\mathcal{E})) \cap \bigoplus_{\gamma \in \Gamma_0} A_\gamma) \bigoplus (d_k(P_k(\mathcal{E})) \cap \bigoplus_{\gamma \in \Gamma_1} A_\gamma).$$

Next, we claim that $d_k(P_k(\mathcal{E})) \cap \bigoplus_{\gamma \in \Gamma_0} A_\gamma = d_k(P_k(\mathcal{F}))$. In fact, if $d \in d_k(P_k(\mathcal{F}))$, then $d = \Sigma_{j=1}^m r_j x_{\gamma_j}$ where, by definition, $\gamma_j \in \Gamma_0$. Hence,

$$d_k(P_k(\mathcal{F})) \subseteq d_k(P_k(\mathcal{E})) \cap \bigoplus_{\gamma \in \Gamma_0} A_\gamma.$$

Now let $d \in d_k(P_k(\mathcal{E})) \cap \bigoplus_{\gamma \in \Gamma_0} A_\gamma$. Thus,

$$d = \sum_{i=1}^{m} a_i < e_0^i, \ldots, e_{k-1}^i >= \sum_{j=1}^{t} r_j x_{\gamma_j} = \sum_{j=1}^{t} r_j (\sum_{l_j=1}^{s_j} c_{l_j} < e_0^{l_j}, \ldots, e_{k-1}^{l_j} >);$$

as $\gamma_j \in \Gamma_0$ for every $j$, $1 \leq j \leq t$, we have $\mathcal{H} = \bigcup_{j=1}^{t}(\bigcup_{l_j=1}^{s_j}\{e_0^{l_j}, ..., e_{k-1}^{l_j}\}) \in [\mathcal{F}]^{<\omega}$. Then, letting $e = s(\mathcal{H})$, we have

$$d_k(e^*(d)) = d - e^*(d_{k-1}(d)) = d,$$

where

$$e^*(d) = \sum_{j=1}^{t} r_j(\sum_{l_j=1}^{s_j} c_{l-j} < e, e_0^{l_j}, ..., e_{k-1}^{l_j} >) \in P_k(\mathcal{F}).$$

Therefore $d \in d_k(P_k(\mathcal{F}))$, and so the claim is verified. It follows that there exists an $A$-module, $M$, with

$$d_k(P_k(\mathcal{E})) = d_k(P_k(\mathcal{F})) \bigoplus M,$$

hence, the theorem is proved. □

**Remark 7.1.4** The finitary function $(\cdot)'$ defined above was first used by Kaplansky in [Ka].

## 7.2 The Main Theorem

In order to apply the previous results to von Neumann regular $f$-rings, a lemma is stated which supplies a correspondence between well ordered chains in $\mathfrak{B}(Max(A))$ and well ordered chains of idempotents in $A$.

**Lemma 7.2.1** *Let $A$ be a von Neumann $f$-ring and $\mathcal{C}$ be a well ordered chain under $\subseteq$ in $\mathfrak{B}(Max(A))$. Then there exists collection, $\mathcal{E} = \{e_C\}_{C \in \mathcal{C}}$, of idempotents of $A$, well ordered under the $f$-ring order on $A$, such that the well ordering on $\mathcal{E}$ is compatable with the given well ordering on $\mathcal{C}$.*

**Proof:** A direct application of Stone duality. □

**Remark 7.2.2** We note that the converse of 7.2.1 is also true, again by a direct application of Stone duality.

We now employ 7.1.2 and 7.1.3 to compute $pd(I_{\mathcal{E}})$, for a linearly ordered collection of idempotents of $A$, which is, in some sense, a minimal generating set for $I_{\mathcal{E}}$.

**Theorem 7.2.3** *Let $A$ be a von Neumann regular $f$-ring with $\mathcal{E}$ a collection of idempotents of $A$ well ordered by the $f$-ring order on $A$, where $|\mathcal{E}| = \aleph_n$, and no subset of cardinality strictly less than $\aleph_n$ generates $I_{\mathcal{E}}$. Then $pd(I_{\mathcal{E}}) = n$.*

**Proof:** The proof proceeds by induction on $n$. For $n = 0$, the result is true as any countably gerenated ideal in a von Neumann ring is projective, [DM]. Given the method of proof to follow, we must also show the result is true for $n = 1$. Thus, by way of contradiction, let $|\mathcal{G}| = \aleph_1$ and assume $pd(I_{\mathcal{E}}) = 0$. Then, by 7.2.2, there exists a collection $\mathcal{C} \subseteq \mathfrak{B}(Max(A))$, well ordered under $\subseteq$ with order type $\aleph_1$. By [DM], we have $K = \bigcup_{C \in \mathcal{C}} C$ is a paracompact subspace of $Max(A)$. Since $K$ is a srtictly ascending chain of clopen, hence compact, sets in $Max(A)$, it follows that $K$ is countably compact. Then, by [En], $K$ is also compact. However, as was previously noted, $K$ is a strictly ascending union of clopen, hence open, sets. Therefore, $K$ cannot be compact, and a contradiction is reached. Next, notice that by Auslander's Lemma, $pd(I_{\mathcal{E}}) \leq 1$, and so $pd(I_{\mathcal{E}}) = 1$. Now, assume the result is true for every $k < n$, and let $|\mathcal{E}| = \aleph_n$. Again, Auslander's Lemma provides the necessary upper bound. Thus, assume, by way of contradiction, that $pd(I_{\mathcal{E}}) \leq n - 1$; hence $d_{n-1}(P_{n-1}(\mathcal{E}))$ is projective. By 7.1.3, there exists $\mathcal{F} \subseteq \mathcal{E}$ such that $|\mathcal{F}| = \aleph_{n-1}$, no subset of cardinality strictly less than $\aleph_{n-1}$ generates $I_{\mathcal{F}}$, and $d_{n-1}(P_{n-1}(\mathcal{F}))$ is a summand of $d_{n-1}(P_{n-1}(\mathcal{E}))$. We claim that, if $e \in \mathcal{E}$, such that $e > e'$, for every $e' \in \mathcal{F}$, then

$$d_{n-1}(e^*(P_{n-2}(\mathcal{F}))) = P_{n-2}(\mathcal{F}) \bigoplus e^*(d_{n-2}(P_{n-2}(\mathcal{F}))),$$

and there exists an $A$-module, $Q$, such that

$$d_{n-1}(e^*(P_{n-2}(\mathcal{F}))) \bigoplus Q = P_{n-2}(\mathcal{E}).$$

In fact, the first equality follows from (a) of the proof of 7.1.2; the second follows as well from (a) of the proof of 7.1.2 and from the decomposition

$$P_{n-2}(\mathcal{E}) = P_{n-2}(\mathcal{F}) \bigoplus (\bigoplus_{\exists e_i \notin \mathcal{F}} A < e_0, ..., e_i, ..., e_{n-2} >).$$

From the above equalities, we have that

$$e^*(d_{n-2}(P_{n-2}(\mathcal{F}))) \bigoplus P_{n-2}(\mathcal{F}) \bigoplus Q = P_{n-2}(\mathcal{C}),$$

hence $e^*(d_{n-2}(P_{n-2}(\mathcal{F})))$ is projective. Noting that

$$e^*(d_{n-2}(P_{n-2}(\mathcal{F}))) \cong d_{n-2}(P_{n-2}(\mathcal{F})),$$

we conclude that $d_{n-2}(P_{n-2}(\mathcal{F}))$ is projective, i.e., $pd(I_{\mathcal{F}}) \leq n-2$, which contradicits the induction hypothesis, as $|\mathcal{F}| = \aleph_{n-1}$. Therefore, $pd(I_{\mathcal{E}}) = n$. □

**Corollary 7.2.4** *Let $A$ be a von Neumann $f$-ring such that $dep(Max(A)) = \aleph_n$. Then $gD(A) \geq n+1$.*

**Proof:** Since $dep(Max(A)) = \aleph_n$, by 7.2.1 there exists collection, $\mathcal{E}$, of idempotents of $A$ of cardinality $\aleph_n$ satisfying the hypotheses of 7.2.3. Thus, the ideal $I_{\mathcal{E}}$ has projective dimension $n$. Invoking 5.1.4, we conclude that $gD(A) \geq n+1$. □

The last step before the statement and proof of the promised result, is a lemma regarding the relationship between the global dimensions of a von Neumann $f$-ring, $A$, and a ring of quotients of $A$ via a multiplicative system. For a review of this concept, the reader is reffered to [AM]. For a justification of the following lemma, the reader is directed to [R], 3.73.

**Lemma 7.2.5** *Let $A$ be a ring and $S$ a multiplicative system of $A$. Then $gD(S^{-1}A) \leq gD(A)$.*

**Proposition 7.2.6** *Let $X$ be a P-space and $K \subseteq \beta X$, with $K$ closed. If $\pi : C(X) \longrightarrow C(K \cap X)$ is the restriction map, then:*

*(a) There exists a multiplicative system, $S \subseteq C(X)$, such that $S^{-1}C(X) \cong \pi(C(X))$.*

*(b) $K \cong Max(S^{-1}(C(X)))$.*

**Proof:** To prove (a), define $S = \{g \in C(X) : coz(g) \supseteq K \cap X\}$. Then, as $coz(g) \cap coz(f) = coz(gf)$, it follows that $S$ is a multiplicative system. For, $\phi : C(X) \longrightarrow S^{-1}C(X)$, the canonical ring of quotients map, we have

$$ker(\phi) = \{ f \in C(X) : \exists g \in S, \text{such that } gf = 0 \}$$

$$= \{ f \in C(X) : Z(f) \supseteq K \} = ker(\phi).$$

It follows that

$$\pi(C(X)) \cong C(X)/ker(\pi) \cong C(X)/ker(\phi) \cong S^{-1}C(X).$$

As for statement (b), since $\pi : C(X) \longrightarrow C(X)/ker(\pi)$ is a surjection, there exists a continuous injection $\pi^* : Max(C(X)/ker(\pi) \longrightarrow Max(C(X)) \cong \beta X$, with

$$Max(\pi(C(X))) \cong Max(C(X)/ker(\pi)) \cong \pi^*(Max(C(X)/ker(\pi)))$$

$$\cong \{ \mathrm{m} \in Max(C(X)) : \mathrm{m} \supseteq ker(\pi) \} \cong K.$$

Thus, by the isomorphism in (a), we have $K \cong Max(S^{-1}C(X))$, and the proof is complete. □

Finally, the promised result.

**Theorem 7.2.7** *Let $X$ be a P-space where $t(\beta X) = \aleph_n$. Then*

$$gD(C(X)) \geq n + 1.$$

**Proof:** Since $\aleph_n = t(\beta X) = hcdep(\beta X)$, there exists $K$, a closed subspace of $\beta X$, such that $dep(K) = \aleph_n$. By 7.2.6, there exists $S$, a multiplicative system of $C(X)$, such that $K \cong Max(S^{-1}C(X))$. As $S^{-1}C(X)$ is a von Neumann regular $f$-ring, by 7.2.4, $gD(S^{-1}C(X)) \geq n+1$. Therefore, invoking 7.2.5, we conclude $gD(C(X)) \geq gD(S^{-1}C(X)) \geq n+1$. □

**Example 7.2.8** As promised in 6.1.12, we now supply an example of a space, $X$, in $\mathcal{KZH}$, such that $gD(\mathfrak{B}(X)) = n+1$, but $ind(X) \neq \aleph_n$. In fact, for the space which we consider, $\mathfrak{B}(X)$ has no infinite, independent sets whatsoever.

The space is formed by taking the one point compactification of $\omega_n$, the $n^{th}$ uncountable ordinal under the order topology. We denote this space by $\omega_n^*$. To see that $\mathfrak{B}(\omega_n^*)$ has no infinite, independent sets, note that every closed subspace of $\omega_n^*$ has an isolated point. Then, by [Ko], 10.19, $\mathfrak{B}(\omega_n^*)$ has no infinite, independent sets. On the other hand, considering any cofinal sequence of isolated points in $\omega_n$, we conclude $dep(\omega_n^*) = \aleph_n$. Invoking 7.2.4, it follows that $gD(\mathfrak{B}(X)) = n+1$.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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